Ph 106c 2004

MIDTERM EXAM SOLUTIONS

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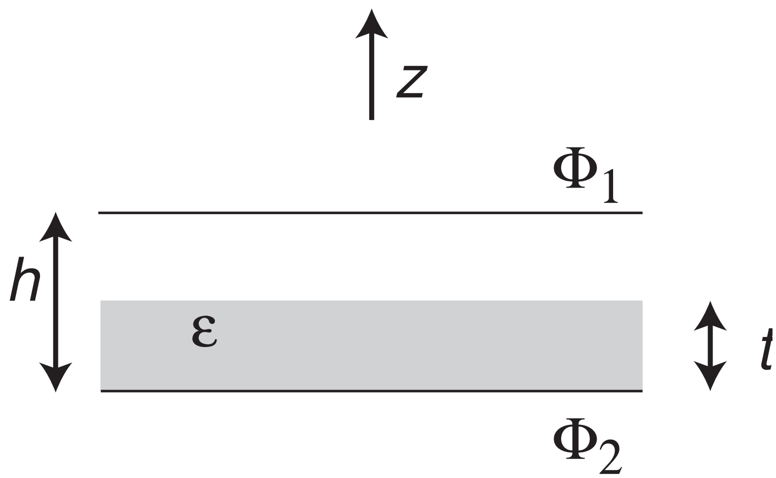

Figure 1: Dielectric slab in between two conducting plates. The $\hat{z}$ direction is normal to the plates.

Problem 1 (15 points total)

Two parallel conducting plates, each with area A, are separated by a distance h and are held at potentials Φ_1 and Φ_2. A uniform slab of linear isotropic dielectric material with $\epsilon = \epsilon_r \epsilon_0$ and thickness t is placed on the bottom plate (see Fig. 1). Ignore fringing fields.

(4 points) **(a)** What is the capacitance ? What is the limiting value as $\epsilon \to \infty$? Is it what you expect ?

Let region I be the empty region $t < z < h$, and region II be the filled region $0 < z < t$. The fields in these regions will be denoted by $\vec{E}_I, \vec{D}_I, \vec{E}_{II}$, and $\vec{D}_{II}$. Except on the capacitor plates, there is no free charge. From $\vec{\nabla} \cdot \vec{D} = 0$, we know that $\hat{z} \cdot \vec{D}$ is equal on the two sides of the $z = t$ boundary. Since the fields point in the $-\hat{z}$ direction, we write

$$\vec{D}_I = \vec{D}_{II} = -D_I \hat{z} = D_{II} \hat{z} \ .$$

The surface charge density on the plates can be obtained by applying the divergence theorem to $\vec{\nabla} \cdot \vec{D} = \rho$ (this just yields Gauss' law); the result is $\sigma_1 = D_I$ and $\sigma_2 = -D_{II}$. The voltage between the plates is

$$V = \Phi_1 - \Phi_2 = \int_1^2 \vec{E} \cdot d\vec{l} = E_I(h-t) + E_{II}t = \frac{1}{\epsilon_0} D_I(h-t) + \frac{1}{\epsilon_r \epsilon_0} D_{II} t \ .$$

Using $D_I = D_{II} = \sigma_1 = \sigma_2 = Q/A$, we have

$$V = \frac{1}{\epsilon_0} \left[h - t + \frac{t}{\epsilon_r} \right] \frac{Q}{A}$$

and so the capacitance is

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{h - t + t/\epsilon_r} \ .$$

As $\epsilon_r \to \infty$, $C \to \epsilon_0 A/(h-t)$, as we expect if the dielectric is replaced by a perfect conductor.

(3 points) **(b)** Suppose (for this part only !) that the dielectric is replaced by a *ferro-electric* material, which has a constant polarization $\vec{P} = P\hat{z}$ independent of $\vec{E}$. For $\Phi_1 = V_1$, $\Phi_2 = 0$, calculate $\vec{E}$, $\vec{D}$, and Φ everywhere between the plates.

We can immediately write

$$\vec{D}_{II} = -\epsilon_0 \vec{E}_{II} \hat{z} + P\hat{z} = -\vec{D}_{II}\hat{z}$$

so

$$E_{II} = \frac{1}{\epsilon_0}(D_{II} + P) \ .$$

The voltage between the plates is now

$$V_1 = E_I(h-t) + E_{II}t = \frac{h-t}{\epsilon_0} D_I + \frac{t}{\epsilon_0} D_{II} \ .$$

Since $D_I = D_{II}$ still holds, we can solve for

$$\vec{D}_I = \vec{D}_{II} = -\left[\frac{\epsilon_0 V_1}{h} - \frac{t}{h} P \right] \hat{z},$$

and the corresponding electric fields are

$$\vec{E}_I = \frac{1}{\epsilon_0} \vec{D}_I = -\left[\frac{V_1}{h} - \frac{t}{h} \frac{P}{\epsilon_0} \right] \hat{z},$$

while

$$\vec{E}_{II} = \frac{1}{\epsilon_0} \left[\vec{D}_{II} - \vec{P} \right] = -\left[\frac{V_1}{h} + \frac{h-t}{h} \frac{P}{\epsilon_0} \right] \hat{z} \ .$$

Since $\vec{E} = -\vec{\nabla}\Phi$, the potential can be calculated by integrating the electric field:

$$\Phi(z) = -\int_0^z dz' \left[\vec{E}(z') \cdot \hat{z}\right] \ .$$

This gives

$$\Phi(z) = \frac{V_1 z}{h} + \frac{P}{\epsilon_0}\frac{(h - z_>)z_<}{h}$$

where $z_> = \max(z, t)$ and $z_< = \min(z, t)$.

(4 points) **(c)** The plates are grounded, $\Phi_1 = \Phi_2 = 0$. Suppose that a point charge q is inserted at a position $\vec{r}_q$ between the plates, either in the free–space region, $t < z_q < h$, or in the dielectric slab, $0 < z_q < t$. Calculate the total charge Q_1 induced on the top plate for both cases. *Hint*: consider the integral

$$I = \int_V d^3\vec{r}\ \epsilon(\vec{r}) \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right]$$

where V is the volume between the plates. Apply Green's theorem, using a judicious choice for ϕ and ψ.

Since $\epsilon(\vec{r})$ is constant in regions I and II, we may split up the integral

$$I = \epsilon_0 \int_{V_I} d^3\vec{r}\ \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right] + \epsilon \int_{V_{II}} d^3\vec{r}\ \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right] \ .$$

Green's theorem tells us that

$$\int_{V_I} d^3\vec{r}\ \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right] = \int_{S_I} \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] da$$

and a similar expression holds for region II. Note that S_I includes the top plate at $z = h$ as well as the boundary between the regions, $z = t$. Meanwhile, S_{II} includes the bottom plate at $z = 0$ as well as the $z = t$ boundary. Because $\vec{D} \cdot \hat{z}$ is conserved at the $z = t$ boundary, and because $\hat{n} = -\hat{z}$ for S_I while $\hat{n} = +\hat{z}$ for S_{II} on this boundary, we see that

$$\epsilon_0 \left.\frac{\partial\psi}{\partial n}\right|_{S_I} = -\epsilon \left.\frac{\partial\psi}{\partial n}\right|_{S_{II}}$$

and similarly for ϕ. Also, both ψ and ϕ are continuous across the $z = t$ boundary. We therefore conclude that the two integrals on the $z = t$ boundary cancel each other, so that

$$I = \epsilon_0 \int_{\text{top plate}} \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] da + \epsilon \int_{\text{bottom plate}} \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] da \ .$$

Furthermore, the surface charge densities on the plates are given by $\sigma = -\vec{D} \cdot \hat{z}$ for the top plate and $\sigma = \vec{D} \cdot \hat{z}$ for the bottom plate. Relating $\vec{D} \cdot \hat{z}$ to the normal derivatives of the potential gives

$$I = \int_{\text{top plate}} \left[\phi\sigma_\psi - \psi\sigma_\phi\right] da + \int_{\text{bottom plate}} \left[\phi\sigma_\psi - \psi\sigma_\phi\right] da \ .$$

We choose ψ to be the potential when $\Phi_1 = V_1$, $\phi_2 = 0$, and no free charge exists between the plates, so that $\nabla^2\psi = 0$. Meanwhile, ϕ is taken to be the potential when $\Phi_1 = \Phi_2 = 0$, and a point charge is placed between the plates:

$$\nabla^2\phi = -\frac{q}{\epsilon(\vec{r}_q)}\delta(\vec{r} - \vec{r}_q) \ .$$

Thus

$$I = \int_V d^3\vec{r}\,\epsilon(\vec{r})\left[\phi\nabla^2\psi - \psi\nabla^2\phi\right] = \int_V d^3\vec{r}\,\epsilon(\vec{r})\psi(\vec{r})\frac{q}{\epsilon(\vec{r})}\delta(\vec{r} - \vec{r}_q)$$

so

$$I = q\,\psi(\vec{r}_q) \ .$$

Meanwhile, most of the surface integrals vanish, leaving only

$$I = -\int_{\text{top plate}} \psi\sigma_\phi da = -V_1\int_{\text{top plate}} \sigma_\phi da = -V_1\,Q_1$$

where Q_1 is the charge induced on the top plate by the point charge at $\vec{r}_q$. We have therefore shown that

$$Q_1 = -q\,\frac{\psi(\vec{r}_q)}{V_1} \ .$$

All that is left is to determine the potential ψ, which is easily done by integrating the electric field from part (a):

$$\frac{\psi(\vec{r}_q)}{V_1} = \frac{z_<}{\epsilon_r(h-t)+t} + \frac{z_> - t}{h - t + t/\epsilon_r} \ .$$

Here, $z_> = \max(z_q, t)$ and $z_< = \min(z_q, t)$. You can check this result by examining the cases $\epsilon_r \to 1$ and $\epsilon_r \to \infty$ (try this as an exercise).

(4 points) **(d)** Suppose that in addition to the usual linear response of the polarization to the electric field, the dielectric slab between plates also has a *non–uniform* built–in polarization density $\delta\vec{P}(\vec{r})$, so that

$$\vec{P}(\vec{r}) = \epsilon_0\chi_e\vec{E} + \delta\vec{P}(\vec{r}) \ .$$

Obtain an expression for the charge δQ_1 on the top plate that is induced by the built–in polarization $\delta\vec{P}(\vec{r})$.

The key for this problem is to remember that the polarization $\delta\vec{P}(\vec{r})$ tells us the additional dipole moment per unit volume, and is equivalent to adding some charge density distribution inside the dielectric. Furthermore, we already know the response to a point charge from part (c), so we can use superposition to get the answer.

Using the definition of $\vec{D}$,

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon_0\vec{E} + \epsilon_0\chi_e\vec{E} + \delta\vec{P} \ .$$

Since there are no free charges,

$$0 = \vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot (\epsilon \vec{E}) + \vec{\nabla} \cdot \delta \vec{P} \ ,$$

defining $\epsilon = \epsilon_0(1 + \chi_e)$ as usual. Thus, the situation is entirely equivalent to having an ordinary dielectric material, obeying $\vec{D} = \epsilon \vec{E}$, in which a charge density $\delta\rho = -\vec{\nabla} \cdot \delta \vec{P}$ is embedded. Note that $\delta \vec{P}$ and $\delta\rho$ are functions of position inside the material, and are not constant in general.

In part (c), we essentially found the "Green's function" for this problem, namely the total charge on the top plate induced by a point charge somewhere between the plates. Thus, we can immediately use superposition to obtain the desired result,

$$\delta Q_1 = + \frac{1}{\epsilon_r(h-t) + t} \int_{V_{II}} d^3\vec{r}\, z(\vec{\nabla} \cdot \delta \vec{P}) \ .$$

Integration by parts may be used to simplify the result to

$$\delta Q_1 = - \frac{1}{\epsilon_r(h-t) + t} \int_{V_{II}} d^3\vec{r}\, (\hat{z} \cdot \delta \vec{P}) \ .$$

This makes sense; only dipole moments in the $\hat{z}$ direction are effective at inducing charge on the top plate.

Problem 2 (15 points total)

A sphere of radius a is made from a material with dielectric constant $\epsilon = \epsilon_0$ and magnetic permeability $\mu = \mu_0$ and carries a uniform charge density σ on its surface. The sphere spins about a central axis at angular frequency ω.

(5 points) **(a)** Suppose you were interested in measuring the magnetic field produced by the spinning sphere in the laboratory. How difficult would this be ? Make a rough order-of-magnitude calculation of the magnetic field strength B near the surface of the spinning sphere. Assume that the sphere has a radius $a = 10$ cm, the spin rate is $\omega/2\pi = 1000$ revolutions per minute, and that the electrostatic potential of the sphere is around 1 kilovolt. Compare your result with the strength of earth's magnetic field, which is around 0.3 Gauss.

The easiest way to get an answer is to find the equivalent surface current $\vec{K}$ at the equator. To find its magnitude K, consider a small dl perpendicular to the equator (i.e. in the $\hat{\theta}$ direction) and calculate the charge that passes in a time interval dt :

$$dQ = K\,dl\,dt = \sigma\,dl\,(v\,dt)$$

where $v = \omega a$ is the velocity of the sphere at the equator. Thus,

$$K = \sigma\,v = \omega\,a\,\sigma\ .$$

Ampere's law provides the boundary condition

$$\hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K}$$

that the magnetic field must satisfy at a current–carrying surface. Therefore, a rough approximation is that $H \sim K$, and $B \sim \mu_0 K$.

Next, we need to relate the potential to the surface charge density. Recall that

$$\Phi(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

for a uniformly charged sphere. Therefore,

$$\sigma = \frac{Q}{4\pi a^2} = \frac{\epsilon_0 \Phi(a)}{a}\ ,$$

where $\Phi(a) = 1$ kV.

Putting everything together:

$$B \sim \mu_0\,\epsilon_0\,\omega\,\Phi(a) \approx\ 1.2 \times 10^{-12}\ \text{Tesla} = 1.2 \times 10^{-8}\ \text{Gauss}\ .$$

This is tiny compared to the Earth's field of ~ 0.3 Gauss. Nonetheless, Henry Rowland succeeded in performing a similar experiment, detecting the magnetic field produced by a charged spinning disk, back in 1876!

(10 points) (b) Obtain expressions for the vector potential $\vec{A}(\vec{r})$ and the magnetic field $\vec{B}(\vec{r})$, both inside and outside the sphere.

OK, so this is where you need to use some heavy math. One way to proceed is to calculate the vector potential $\vec{A}$, and then use $\vec{B} = \vec{\nabla} \times \vec{A}$. The vector potential satisfies

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

provided we are working in the Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$. The formal solution is provided by using the free–space Green's function,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3\vec{r}\,' \, \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \; .$$

The current density is nonzero only on the surface of the sphere, so

$$\vec{J}(\vec{r}\,') = \sigma \omega a \sin\theta \delta(r' - a)\, \hat{\phi}\,' \; .$$

It is important to remember that $\hat{\phi}\,'$ will not be a constant inside the integral – it really should be written as

$$\hat{\phi}\,' = -\sin\phi' \hat{x} + \cos\phi' \hat{y}$$

and so cannot simply be pulled out of the integral. Nonetheless, symmetry about the rotation axis tells us that the resulting vector potential must be in the direction $\hat{\phi}$.

The rest is grunge. You need to use the spherical harmonic expansion

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\Omega')\, Y_{lm}(\Omega) \; .$$

Also, using $Y_{1\pm 1} \sim \sin\theta\, e^{\pm i\phi}$, you can write

$$\vec{J}(\vec{r}\,') = \sqrt{\frac{8\pi}{3}}\, \sigma\omega a \delta(r' - a) \left\{ \frac{1}{2i} \left[Y_{11}(\Omega') + Y_{1-1}(\Omega') \right] \hat{x} - \frac{1}{2} \left[Y_{11}(\Omega') - Y_{1-1}(\Omega') \right] \hat{y} \right\} \; .$$

The only thing left to do is integrate. The orthogonality of the spherical harmonics kills all terms except $l = 1$, $m = \pm 1$. The radial integral is easy because of the delta function. After some straightforward algebra (do this as an exercise), one finds

$$\vec{A}(\vec{r}) = \hat{\phi}\, \frac{\mu_0 \sigma \omega a^2}{3} \sin\theta \begin{cases} r/a, & r < a \\ (a/r)^2, & r > a \end{cases} \; .$$

To calculate $\vec{B}$, we can look up the curl in spherical coordinates and grind away. This works, is pretty quick, but not so illuminating.

Another way to proceed is to realize that $r\sin\theta\hat{\phi} = \hat{z}\times\vec{r}$. This means that the vector potential we found for the $r > a$ region is exactly the same as that produced by a magnetic moment (Jackson 5.55):

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{|\vec{r}|^3} \ ,$$

where the magnetic moment of our spinning sphere is

$$\vec{m} = \frac{1}{3}Q\omega a^2\hat{z} \ .$$

Thus, the field outside the sphere is just a dipole field (Jackson 5.64):

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\frac{3\hat{r}(\hat{r}\cdot\vec{m})-\vec{m}}{|\vec{r}|^3}\right] \ .$$

Just outside the sphere, at the equator, you can show that this exact answer is a factor of 3 weaker than the rough estimate in part (a). (Do this as an exercise).

Inside the sphere, the result is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{a^3} \ .$$

Since

$$\vec{\nabla}\times(\vec{m}\times\vec{r}) = \vec{m}(\vec{\nabla}\cdot\vec{r}) - (\vec{m}\cdot\vec{\nabla})\vec{r} = 3\vec{m}-\vec{m} = 2\vec{m} \ ,$$

we see that the field inside the sphere is uniform:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{2\pi}\frac{\vec{m}}{a^3} \ .$$

To check these results, you can independently calculate the magnetic moment of the spinning sphere, using Jackson (5.54):

$$\vec{m} = \frac{1}{2}\int d^3\vec{r}\ \vec{r}\times\vec{J}(\vec{r})$$

which for a surface current reads

$$\vec{m} = \frac{1}{2}\int_S da\ \vec{r}\times\vec{K}(\vec{r}) \ .$$

Using $\vec{K} = \sigma\omega a\sin\theta\,\hat{\phi}$, one can show that the same result for $\vec{m}$ is obtained. (Try this as an exercise).

This problem is equivalent to the uniformly magnetized sphere that is worked out in Jackson.