Solutions to Problem Set 7.

June 6, 2004

11.14(a)(b)

(a) We will use Eqs.(11.137),(11.138), (11.140), together with

$$\mathcal{F}_{\alpha\beta} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}. \tag{1}$$

The solution is now a matter of simply writing down the various contractions explicitly. In particular,

$$F^{\alpha\beta}F_{\alpha\beta} = -\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B} + \vec{B}\cdot\vec{B} - \vec{E}\cdot\vec{E} = 2(B^2 - E^2) \tag{2}$$

$$\mathcal{F}^{\alpha\beta}F_{\alpha\beta} = -\vec{B}\cdot\vec{E} - \vec{E}\cdot\vec{B} - \vec{E}\cdot\vec{B} - \vec{B}\cdot\vec{E} = -4\vec{E}\cdot\vec{B} \tag{3}$$

$$\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta} = -\vec{B}\cdot\vec{B} + \vec{E}\cdot\vec{E} + \vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B} = 2(E^2 - B^2). \tag{4}$$

There are no other quadratic invariants – we have constructed all the possible nontrivial contractions of two field strength tensors. In particular, notice that $\mathcal{F}^{\alpha\beta}F_{\alpha\beta} = F^{\alpha\beta}\mathcal{F}_{\alpha\beta}$.

(b) Here we use the invariants that we found in (a). In particular, $E^2 - B^2$ is invariant, so if we have fields $\vec{E} = 0$ and $\vec{B}$ in one frame and fields $\vec{E}'$ and $\vec{B}' = 0$ in another, we must have

$$-B^2 = E^2 - B^2 = (E')^2 - (B')^2 = (E')^2, \tag{5}$$

which is impossible unless $B = 0$ and $E' = 0$, since both B^2 and $(E')^2$ are nonnegative.

For the second part, we would like to find the criteria on $\vec{E}$ and $\vec{B}$ which will guarantee the existence of a frame in which the electric field is zero. Again, we appeal to the invariants found in (a).

In particular, we know that $E^2 - B^2$ and $\vec{E} \cdot \vec{B}$ are invariant. So, if there is some frame in which $\vec{E}' = 0$, we must have

$$\vec{E} \cdot \vec{B} = \vec{E}' \cdot \vec{B}' = 0 \tag{6}$$

and

$$E^2 - B^2 = 0 - (B')^2 = -(B')^2. \tag{7}$$

Thus, the condition is that $\vec{E} \cdot \vec{B} = 0$ and $E^2 - B^2 < 0$. Since $E^2 - B^2$ and $\vec{E} \cdot \vec{B}$ are the *only* invariants of $\vec{B}$ and $\vec{E}$, if these conditions are satisfied, there *must* be a frame in which the electric field is zero.

11.20

(a) By conservation of 4-momentum,

$$p_\Lambda = p_1 + p_2, \tag{8}$$

where p_Λ,p_1 and p_2 are the 4-momenta of the Λ-particle, the nucleon and the pi-meson, respectively. Squaring each side gives

$$M^2 = p_\Lambda^2 = (p_1 + p_2)^2 = p_1^2 + 2p_1 \cdot p_2 + p_2^2 = m_1^2 + 2(E_1 E_2 - \vec{p_1} \cdot \vec{p_2}) + m_2^2 \tag{9}$$

$$= m_1^2 + m_2^2 + 2E_1 E_2 - 2|\vec{p_1}||\vec{p_2}| \cos(\theta). \tag{10}$$

(b) How far on average will the Λ travel before decaying? Well, it has energy $10GeV$ and mass $1.115GeV$, so $\gamma = \frac{E}{m} = 8.97$ and $v = .9938c$. Its average lifetime in its rest frame is 2.9×10^{-10} seconds, so in the lab frame its average lifetime is $\gamma 2.9 \times 10^{-10} = 2.6 \times 10^{-9}s$, which means it travels on average $2.6 \times 10^{-9}s \times .9938c = 0.775m$ before decaying.

What is the range of possible angles?

Conservation of 4-momentum requires $p_1 + p_2 = (M, 0, 0, 0)$ in the rest frame of the Λ particle. Writing out the components explicitly and considering the case when the outgoing particles travel in the x direction (perpendicular to the initial direction of the Λ) gives

$$m_1\gamma_1 + m_2\gamma_2 = M \text{ and} \tag{11}$$
$$m_1\gamma_1 v_1 + m_2\gamma_2 v_2 = 0 \tag{12}$$

which, upon using the identity $1 + v^2\gamma^2 = \gamma^2$ and letting $E_1 = m_1\gamma_1$ and $E_2 = m_2\gamma_2$ implies

$$E_1 + E_2 = M \tag{13}$$
$$m_1^2(\gamma_1^2 - 1) = m_2^2(\gamma_2^2 - 1). \tag{14}$$

The second equation can be simplified to

$$E_1^2 - E_2^2 = m_1^2 - m_2^2. \tag{15}$$

Substituting $E_2 = M - E_1$ into this equation gives

$$E_1^2 - (M - E_1)^2 = m_1^2 - m_2^2, \tag{16}$$

which implies

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} = 944MeV \Rightarrow \gamma_1 = \frac{944}{939} = 1.005. \tag{17}$$

Thus,

$$E_2 = 1115 - 944 = 171MeV \Rightarrow \gamma_2 = \frac{171}{140} = 1.22. \tag{18}$$

This allows us to find the magnitudes of the 3-momenta:

$$|\vec{p_1}| = |\vec{p_2}| = m_2 v_2 \gamma_2 = m_2\sqrt{\gamma_2^2 - 1} = 98.2MeV. \tag{19}$$

Now we apply the result of part (a), which implies

$$E_1E_2 - |\vec{p_1}||\vec{p_2}|\cos\phi = E_1'E_2' - |\vec{p_1}'||\vec{p_2}'|\cos\theta, \tag{20}$$

where the primed quantities are in measured in the lab frame. We know that ϕ is π and would like to know the range of values of θ. The left hand side of the equation is

$$E_1E_2 + |\vec{p_1}||\vec{p_2}| = 161424 + 9643(MeV)^2 = 171067(MeV)^2. \tag{21}$$

Since the particles are highly relativistic in the lab frame, the right hand side is well approximated by

$$m_1 m_2 \gamma^2 - m_1 m_2 (\gamma^2 - 1)\cos\theta = 1.058 \times 10^7 - 1.045 \times 10^7 \cos\theta, \qquad (22)$$

which implies

$$\cos\theta = \frac{1.058 \times 10^7 - 1.71 \times 10^5}{1.045 \times 10^7} = 0.9961. \qquad (23)$$

Alternatively, we could use our estimate $E_1' E_2' - |\vec{p_1}'||\vec{p_2}'|\cos\theta \approx m_1 m_2 \gamma^2 - m_1 m_2(\gamma^2 - 1)\cos\theta$ directly with the result of (a) to get the same result without any need to assume the particles move in the x direction in the Λ rest frame.

Thus, for the case where the outgoing particles move perpendicular to the Λ's initial direction, the angle between the two particles in the lab frame is $\cos^{-1}(.9961) = 0.088$ radians (or 5.1 degrees). If the outgoing particles move in any other direction, the result will be a smaller angle.

11.22

Let k_1 and k_2 be the 4-momenta of the initial state photons, and p_1 and p_2 be the 4-momenta of the outgoing electron-positron pair. From the 4-momentum conservation law,

$$(k_1 + k_2)^2 = (p_1 + p_2)^2 \qquad (24)$$

Rewritten in the $3+1$-dimensional form, the right hand side becomes

$$\left(\sqrt{m^2 + \vec{p}_1^2} + \sqrt{m^2 + \vec{p}_2^2}\right)^2 - (\vec{p}_2 + \vec{p}_2)^2 = 2m^2 + 2\sqrt{(m^2 + \vec{p}_1^2)(m^2 + \vec{p}_2^2)} - 2(\vec{p}_1 \cdot \vec{p}_2)$$

where m is the electron mass.

The above function is always greater than $4m^2$ (and attains this minimal value for $\vec{p}_1 = \vec{p}_2 = 0$). For that reason, the equation (24) will have solutions if and only if $(k_1 + k_2)^2 \geq 4m^2$. For $k_1 = E_1(1, \vec{n}_1)$ and $k_2 = E_2(1, \vec{n}_2)$ (with $\vec{n}_1$ and $\vec{n}_2$ being unit vectors),

$$(k_1 + k_2)^2 = (E_1 + E_2)^2 - (E_1 \vec{n}_1 + E_2 \vec{n}_2)^2 = 2E_1 E_2 \left(1 - (\vec{n}_1 \cdot \vec{n}_2)\right) \qquad (25)$$

Given the energy values, the reaction will occur if $\vec{n}_1$ and $\vec{n}_2$ can be chosen so as to make (25) greater than $4m^2$. The maximal value (25) can attain if we're allowed to adjust $\vec{n}_1$ and $\vec{n}_2$ is $4E_1E_2$. The threshold condition thus becomes

$$E_1E_2 \geq m^2$$

For part (a), this will give a photon energy of

$$(5 \times 10^5 eV)^2/(2.5 \times 10^{-4} eV) = 10^{15} eV$$

For part (b),

$$(5 \times 10^5 eV)^2/(5 \times 10^2 eV) = 5 \times 10^8 eV$$