# Solutions to Problem Set 6.

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## 11.6

We will first find the position of a spaceship $x(\tau)$ in the earth frame after a proper time $\tau$ has elapsed. We consider the case of constant proper acceleration $\vec{a} = (a, 0, 0)$ and initial velocity $v(0) = 0$. Now, in the earth frame we have

$$U = (\frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau}) = \gamma(1, \vec{v}), \tag{1}$$

where $\vec{v}$ is the rocket's velocity as measured in the earth frame. The 4-acceleration,

$$A = \frac{dU}{d\tau}, \tag{2}$$

is a 4-vector and thus has an invariant length. That is, $A \cdot A = -(A^0)^2 + \vec{A} \cdot \vec{A}$ has the same value in all inertial frames. If we consider a reference frame with coordinates $\tilde{t}$ and $\tilde{\vec{x}}$ which is instantaneously comoving with the rocket, we find that $\frac{d^2\tilde{t}}{d\tau^2} = 0$ and $\frac{d^2\tilde{\vec{x}}}{d\tau^2} = \vec{a}$, which implies that

$$A \cdot A = \vec{a} \cdot \vec{a} \tag{3}$$

in *all* frames. Combining Eq. (3) with the observation that $U \cdot a = 0$ (which can be obtained by differentiating $U \cdot U = -1$) and Eq. (1), we find that

$$\frac{d^2t}{d\tau^2} = v_x \frac{d^2x}{d\tau^2}, \tag{4}$$

where $t$ and $x$ refer to the earth coordinate system. This shows that the 4-acceleration in the earth frame is given by

$$A = (v_x \frac{d^2x}{d\tau^2}, \frac{d^2x}{d\tau^2}, 0, 0), \quad (5)$$

a fact which can be combined with Eq. (3) to show

$$a^2 = (1 - v^2)(\frac{d^2x}{d\tau^2})^2. \quad (6)$$

Thus, $\frac{d^2x}{d\tau^2} = \gamma a$, which implies together with $\frac{d^2x}{d\tau^2} = \frac{d(\gamma v)}{d\tau} = \gamma^3 \frac{dv}{d\tau}$ that

$$\frac{dv}{d\tau} = a(1 - v^2). \quad (7)$$

Together with the condition that $v(0) = 0$, this equation has the solution $v = \tanh(a\tau)$.

Now, $\frac{dx}{dt} = v(t)$ so that $dx = v(\tau)\gamma(\tau)d\tau$ and thus

$$x(\tau_2) - x(\tau_1) = \int_1^2 dx = \int_{\tau_1}^{\tau_2} v(\tau)\gamma(\tau)d\tau = \int_{\tau_1}^{\tau_2} \frac{\tanh(a\tau)}{\sqrt{1 - \tanh^2(a\tau)}} d\tau \quad (8)$$

$$= \int_{\tau_1}^{\tau_2} \sinh(a\tau)d\tau = \frac{1}{a}\cosh(a\tau_2) - \frac{1}{a}\cosh(a\tau_1). \quad (9)$$

Now we have to reintroduce the factors of $c$ that were suppressed in the calculation above. In particular,

$$x(\tau_2) - x(\tau_1) = \frac{c^2}{a}\cosh(\frac{a\tau_2}{c}) - \frac{c^2}{a}\cosh(\frac{a\tau_1}{c}). \quad (10)$$

This allows us to immediately answer part (b) by taking $\tau_1 = 0$, $\tau_2 = 5$yrs and $a = g$. The result is that in the first 5 years, the spaceship will travel a distance of

$$\frac{c^2}{a}(\cosh(\frac{a\tau_2}{c}) - 1) = c\tau_2(\frac{c}{\tau_2 a})(\cosh(\frac{a\tau_2}{c}) - 1), \quad (11)$$

which by calculating $\frac{a\tau_2}{c} = \frac{9.81 \cdot 5 \cdot 365 \cdot 24 \cdot 3600}{3 \times 10^8} = 5.16$ becomes

$$c \cdot 5\text{yrs}(\frac{1}{5.16})(\cosh(5.16) - 1) = 83.4 \text{ light years} . \quad (12)$$

The distance traveled during the second leg is the same, so that the total distance traveled is 166.8 light years.

We now go back to (a):
Now, $dt = \gamma d\tau$, so we have

$$t = \int_0^\tau \gamma(\tau) d\tau = \int_0^\tau \cosh(a\tau) d\tau = \frac{1}{a}\sinh(a\tau). \tag{13}$$

Again, we must reintroduce the factors of $c$, to get

$$t = \frac{c}{a}\sinh(\frac{a\tau}{c}) = \tau\frac{c}{a\tau}\sinh(\frac{a\tau}{c}) \tag{14}$$

so that the earth time elapsed in the first 5 year leg of the trip is 5 yrs $\frac{1}{5.16}\sinh(5.16) =$ 84.4yrs. The earth time elapsed on each 5 year leg of the trip is the same (which can be seen from the symmetry of the problem), so that the total time elapsed on earth during the entire voyage is $4 \cdot 84.4 = 337.6$years.

## 11.13

a) In the frame $K'$ we simply have an infinite line charge. This has zero magnetic field and a $\frac{1}{r}$ electric field. In particular, we have

$$\begin{aligned} \vec{B}' &= 0 \text{ and} & (15)\\ \vec{E}' &= \frac{2q_0}{r'}\hat{r}'. & (16) \end{aligned}$$

We can transform this back to the lab frame by using the general relation

$$\begin{aligned} \vec{E} &= \gamma(\vec{E}' + \vec{\beta}\times\vec{B}') - \frac{\gamma^2}{1+\gamma}\vec{\beta}(\vec{\beta}\cdot\vec{E}') & (17)\\ \vec{B} &= \gamma(\vec{B}' - \vec{\beta}\times\vec{E}') - \frac{\gamma^2}{1+\gamma}\vec{\beta}(\vec{\beta}\cdot\vec{B}') & (18) \end{aligned}$$

with $\vec{\beta} = -\frac{v}{c}\hat{z}$. In particular, this leads to

$$\begin{aligned} \vec{E} &= \gamma\vec{E}' = \frac{2\gamma q_0}{r'}\hat{r}' \text{ and} & (19)\\ \vec{B} &= \frac{2\gamma q_0}{r'}\frac{v}{c}\hat{z}\times\hat{r}' = \frac{2\gamma q_0 v}{cr'}\hat{\phi} & (20) \end{aligned}$$

b) In $K'$ the charge density is given by

$$\rho(r,z,\theta) = \frac{q_0}{\pi r}\delta(r), \tag{21}$$

which can be seen by noticing that the integral over $r$ and $\theta$ gives the appropriate linear charge density:

$$\int_0^{2\pi}\int_0^{\infty}\rho(r,z,\theta)rdrd\theta = 2\pi\int_0^{\infty}\frac{q_0}{\pi}\delta(r)dr = \pi\int_{-\infty}^{\infty}\frac{q_0}{\pi}\delta(r)dr = q_0. \tag{22}$$

Since $K'$ is the rest frame of the wire, the current density is simply zero. Thus, the 4-current is

$$J = (c\frac{q_0}{\pi r}\delta(r), 0, 0, 0), \tag{23}$$

in the $K'$ frame. Lorentz transforming this into the lab frame gives

$$J = (c\gamma\frac{q_0}{\pi r}\delta(r), v\gamma\frac{q_0}{\pi r}\delta(r)\hat{z}). \tag{24}$$

c) Now we'd like to calculate the $\vec{E}$ and $\vec{B}$ fields directly from Eq. (24). Because of the cylindrical symmetry, the electric field can be determined using Gauss' Law:

$$2\pi rLE(r) = 4\pi\int_0^r\int_0^{2\pi}\int_{L_0}^{L+L_0}\rho(s)dLsdsd\phi = 8\pi^2 L\int_0^r\rho(s)sds = 4\pi L\gamma q_0, \tag{25}$$

leading to $\vec{E}(r') = \frac{2\gamma q_0}{r'}\hat{r}'$, as found in part (a).

To find the magnetic field, we use Ampere's law along a circle of radius $r$ to find

$$\oint\vec{B}\cdot d\vec{l} = \frac{4\pi}{c}\int\vec{J}\cdot d\vec{a}, \tag{26}$$

which implies

$$\begin{aligned}2\pi rB(r) &= \frac{4\pi}{c}\int_0^{2\pi}\int_0^r v\gamma\frac{q_0}{\pi s}\delta(s)sdsd\phi && (27)\\ &= \frac{8\pi^2}{c}v\gamma\frac{q_0}{\pi}\int_0^r\delta(s)ds && (28)\\ &= \frac{8\pi^2}{c}v\gamma\frac{q_0}{\pi}\frac{1}{2}. && (29)\end{aligned}$$

From this we can deduce

$$\vec{B} = \frac{2\gamma v q_0}{cr}\vec{\phi}, \tag{30}$$

again in agreement with (a).

## 12.3

**Warning: we take $c = 1$ for this problem.**

We take $\vec{E_0}$ to point along the $z$ axis and $\vec{v}_0$ along the $x$ axis. Since $\vec{B} = 0$, the equations of motion of the particle are given by

$$\frac{d\vec{p}}{dt} = e\vec{E_0} \tag{31}$$

$$\frac{d\epsilon}{dt} = e\vec{v}\cdot\vec{E_0}, \tag{32}$$

where we have let $\epsilon$ be the energy to avoid confusion with the electric field. Writing these out in component form, we find

$$\frac{dp_x}{dt} = 0 \tag{33}$$

$$\frac{dp_y}{dt} = 0 \tag{34}$$

$$\frac{dp_z}{dt} = eE_0 \tag{35}$$

$$\frac{d\epsilon}{dt} = ev_z E_0, \tag{36}$$

the first three of which have solutions

$$p_x = \text{constant} = 0 \tag{37}$$

$$p_y = \text{constant} = m\frac{v_0}{\sqrt{1-v_0^2}} \tag{38}$$

$$p_z = eE_0 t + \text{constant} = eE_0 t. \tag{39}$$

Now,

$$\epsilon(t) = \sqrt{\vec{p}^2 + m^2} = \sqrt{\frac{m^2 v_0^2}{1-v_0^2} + (eE_0 t)^2 + m^2} = \sqrt{\frac{m^2}{1-v_0^2} + (eE_0 t)^2}, \tag{40}$$

so that

$$\vec{v}(t) = \frac{1}{m\gamma}\vec{p}(t) = \frac{1}{\epsilon}(0, m\frac{v_0}{\sqrt{1-v_0^2}}, eE_0 t) = \frac{1}{\sqrt{\frac{m^2}{1-v_0^2} + (eE_0 t)^2}}(0, m\frac{v_0}{\sqrt{1-v_0^2}}, eE_0 t), \tag{41}$$

which can be integrated to give

$$x(t) = 0 \tag{42}$$

$$y(t) = \frac{m\gamma_0 v_0}{eE_0}\sinh^{-1}(\frac{eE_0 t}{m\gamma_0}) \tag{43}$$

$$z(t) = \sqrt{t^2 + \frac{m^2\gamma_0^2}{(eE_0)^2}} - \frac{m\gamma_0}{eE_0}. \tag{44}$$

(b) Eliminating $t$ in the equations above, we find

$$\sinh(\frac{eE_0}{m\gamma_0 v_0}y) = \frac{eE_0 t}{m\gamma_0} \Rightarrow \tag{45}$$

$$z = \frac{m\gamma_0}{eE_0}[\sqrt{\sinh^2(\frac{eE_0}{m\gamma_0 v_0}y) + 1} - 1] = \frac{m\gamma_0}{eE_0}[\cosh(\frac{eE_0}{m\gamma_0 v_0}y) - 1]. \tag{46}$$

For times $t \ll \frac{m\gamma_0}{eE_0}$ we have $y \ll \frac{m\gamma_0 v_0}{eE_0}$ so that

$$z \approx \frac{m\gamma_0}{eE_0}[1 + \frac{1}{2}(\frac{eE_0}{m\gamma_0 v_0}y)^2 - 1] = \frac{eE_0}{2m\gamma_0 v_0^2}y^2. \tag{47}$$

For $t \gg \frac{m\gamma_0}{eE_0}$ we have

$$z \approx \frac{m\gamma_0}{2eE_0}\exp(\frac{eE_0}{m\gamma_0 v_0}y). \tag{48}$$