Solutions to Problem Set 4.

Graeme Smith

May 21, 2004

5.26

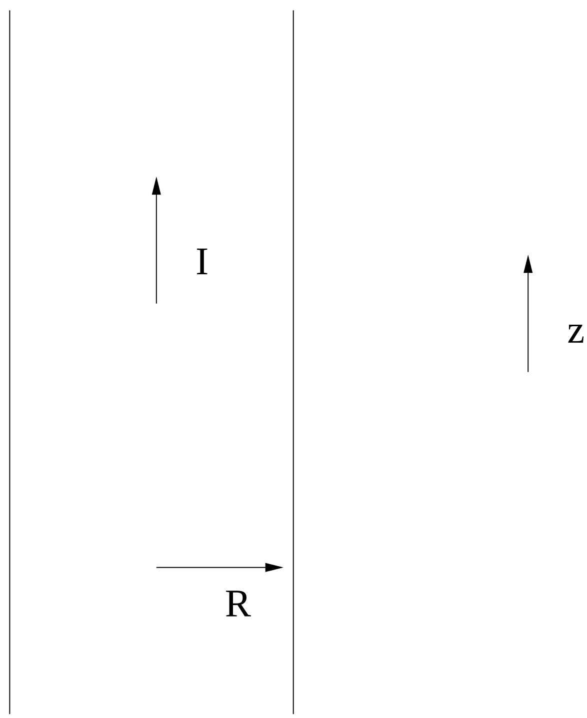

Figure 1:

To find the magnetic field due to a single wire, we use Ampere's law:

Letting ρ be the radial distance to the wire, on the outside of the wire we find

$$2\pi\rho B(\rho) = \int \vec{B} \cdot d\vec{l} = \mu_0 I, \tag{1}$$

which implies

$$\vec{B}_{out}(\rho) = \frac{\mu_0 I}{2\pi\rho}\hat{z}. \tag{2}$$

On the inside, we get

$$2\pi\rho B(\rho) = \int \vec{B} \cdot d\vec{l} = \mu_0 I \frac{\rho^2}{R^2}, \tag{3}$$

where R is the radius of the wire. Using the relation $\vec{\bigtriangledown} \times \vec{A} = \vec{B}$, we find that $B_z = -\frac{\partial}{\partial\rho} A_z$, which implies

$$A_z = -\int B_z d\rho. \tag{4}$$

So,

$$A_z = -\frac{\mu_0 I}{4\pi}(\log(\frac{\rho^2}{R^2}) + 1) \qquad \rho \geq R \tag{5}$$

$$= -\frac{\mu_0 I}{4\pi}\frac{\rho^2}{R^2} \qquad \rho < R. \tag{6}$$

In the system we consider there are two wires. If the wires are both of length l, we know that the total potential energy is given by

$$W = \frac{1}{2}\int \vec{J} \cdot \vec{A} d\vec{x} = \frac{1}{2}\int \vec{J^a} \cdot \vec{A^a} d\vec{x^a} + \frac{1}{2}\int \vec{J^b} \cdot \vec{A^b} d\vec{x^b} \tag{7}$$

$$= \frac{l}{2}(\frac{I}{\pi a^2})\int_0^{2\pi}\int_0^a A_z^{\text{inside a}}(\rho_a) d\rho_a d\phi_1 + \frac{l}{2}(\frac{I}{\pi b^2})\int_0^{2\pi}\int_0^b A_z^{\text{inside b}}(\rho_b) d\rho_b d\phi_1 \tag{8}$$

$$= \frac{l}{2}\frac{I}{\pi a^2}\frac{\mu_0 I}{4\pi}\int_0^{2\pi}\int_0^a [\log\frac{\rho_b^2}{b^2} + 1 - \frac{\rho_a^2}{a^2}] d\rho_a d\phi_1 + \frac{l}{2}\frac{I}{\pi b^2}\frac{\mu_0 I}{4\pi}\int_0^{2\pi}\int_0^b [\log\frac{\rho_a^2}{a^2} + 1 - \frac{\rho_b^2}{b^2}] d\rho_b d\phi_2 \tag{9}$$

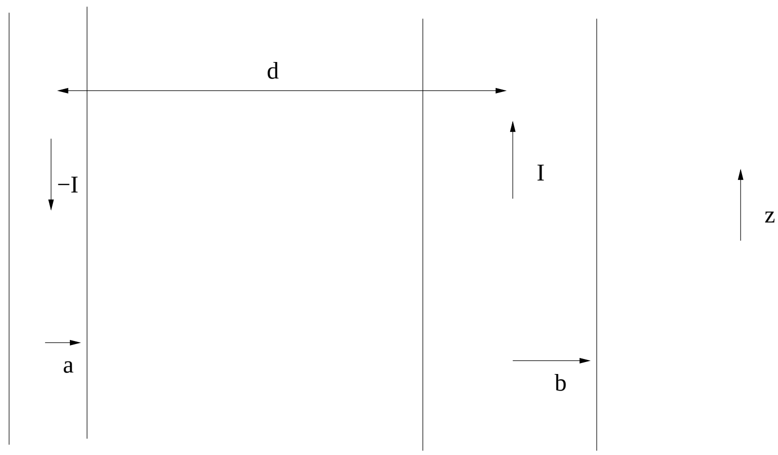

Figure 2:

$$= \frac{l}{2}\frac{I}{\pi a^2}\frac{\mu_0 I}{4\pi}\int_0^{2\pi}\int_0^{a}[\log\frac{\rho_a^2+d^2-2d\rho_a cos\phi_1}{b^2}+1-\frac{\rho_a^2}{a^2}]d\rho_a d\phi_1 \tag{10}$$

$$+\frac{l}{2}\frac{I}{\pi b^2}\frac{\mu_0 I}{4\pi}\int_0^{2\pi}\int_0^{b}[\log\frac{\rho_b^2+d^2-2d\rho_b cos\phi_2}{b^2}+1-\frac{\rho_b^2}{b^2}]d\rho_b d\phi_2 \tag{11}$$

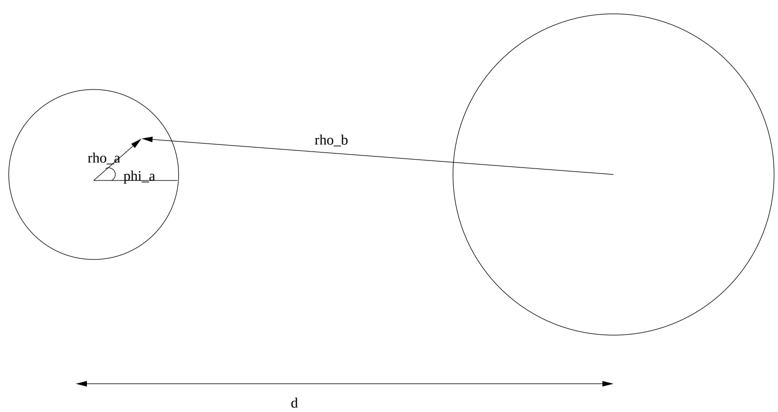

Figure 3:

The most challenging integrals to perform are those of the form

$$\frac{1}{b^2}\int_0^{b}\int_0^{2\pi}[\log(\frac{d^2+\rho_b^2-2d\rho_b cos\phi_2}{a^2})]d\phi_2\rho_b d\rho_b. \tag{12}$$

We perform the ϕ_2 integral first, taking advantage of the fact that for $s > 1$

$$\int_0^{2\pi} \log(s - cos\phi)d\phi = 2\pi \log(\frac{1}{2}(s + \sqrt{s^2 - 1})), \tag{13}$$

which can be established via contour integration (as is done in the appendix), Mathematica, or checking a standard table of integrals. This implies

$$\int_0^{2\pi} \log(a^2 + b^2 - 2abcos\phi)d\phi = 2\pi \log(2ab) + \int_0^{2\pi} \log(\frac{a^2 + b^2}{2ab} - cos\phi)d\phi \tag{14}$$

$$= 2\pi \log(2ab) + 2\pi \log(\frac{1}{2}(\frac{a^2 + b^2}{2ab} + \sqrt{(\frac{a^2 + b^2}{2ab})^2 - 1})) \tag{15}$$

$$= 2\pi \log(\frac{1}{2}(a^2 + b^2 + \sqrt{(a^2 + b^2)^2 - 4a^2b^2})) \tag{16}$$

$$= 2\pi \log(\frac{1}{2}(a^2 + b^2 + \sqrt{(a^2 - b^2)^2})) \tag{17}$$

$$= 2\pi \log(\max(a^2, b^2)). \tag{18}$$

So,

$$\frac{1}{b^2} \int_0^b \int_0^{2\pi} [\log(\frac{d^2 + \rho_b^2 - 2d\rho_b cos\phi_2}{a^2})] d\phi_2 \rho_b d\rho_b \tag{19}$$

$$= \frac{2\pi}{b^2} \int_0^b \log(\frac{d^2}{a^2}) \rho_b d\rho_b \tag{20}$$

$$= 2\pi \log(\frac{d}{a}) \tag{21}$$

A similar argument shows that

$$\frac{1}{b^2} \int_0^a \int_0^{2\pi} [\log(\frac{d^2 + \rho_a^2 - 2d\rho_a cos\phi_1}{b^2})] d\phi_1 \rho_a d\rho_a \tag{22}$$

$$= 2\pi \log(\frac{d}{b}). \tag{23}$$

Using these integrals, we find that

$$W = \frac{l}{2}\frac{\mu_0}{4\pi}(\frac{1}{2} + 2\log\frac{d}{b})I^2 + \frac{l}{2}\frac{\mu_0}{4\pi}(\frac{1}{2} + 2\log\frac{d}{a})I^2 \tag{24}$$

$$= \frac{l}{2}\frac{\mu_0}{4\pi}(1 + 2\log\frac{d^2}{ab})I^2 = \frac{l}{2}(\frac{L}{2})I^2, \tag{25}$$

which implies

$$\frac{L}{l} = \frac{\mu_0}{4\pi}(1 + 2\log\frac{d^2}{ab}). \tag{26}$$

5.29

From the discussion at the beginning of section (5.13), we know that both the $\vec{B}$ and $\vec{E}$ fields are in the x, y plane, and that the $\vec{E}$ is paralell to $\hat{n}$ at the surface while $\vec{B}$ is perpendicular. From the fact that $B_z = 0$ and $E_z = 0$, we can show from Maxwell's equations that $\vec{B} = \pm\sqrt{\mu\epsilon}\hat{z} \times \vec{E}$. In particular,

$$\frac{\partial \vec{E}}{\partial z} = \frac{\partial \vec{B}}{\partial t} \tag{27}$$

and

$$\frac{\partial \vec{B}}{\partial z} = -\mu\epsilon\frac{\partial \vec{E}}{\partial t}, \tag{28}$$

together imply that

$$\frac{\partial^2 \vec{E}}{\partial z^2} = -\mu\epsilon\frac{\partial^2 \vec{E}}{\partial t^2} \tag{29}$$

and

$$\frac{\partial^2 \vec{B}}{\partial z^2} = -\mu\epsilon\frac{\partial^2 \vec{B}}{\partial t^2}, \tag{30}$$

which means that

$$\vec{B}(x, y, z, t) = \vec{B^+}(x, y, t - \sqrt{\mu\epsilon}z) + \vec{B^-}(x, y, t + \sqrt{\mu\epsilon}z), \tag{31}$$

and similarly for $\vec{E}$. Substituting these forms into Eq. (27) and (28) shows that $\vec{B} = \pm\sqrt{\mu\epsilon}\hat{z}\times\vec{E}$. The important point is that $\vec{E}(x, y, z, t)$ and $\vec{B}(x, y, z, t)$ have proportional magnitutes but orthogonal directions.

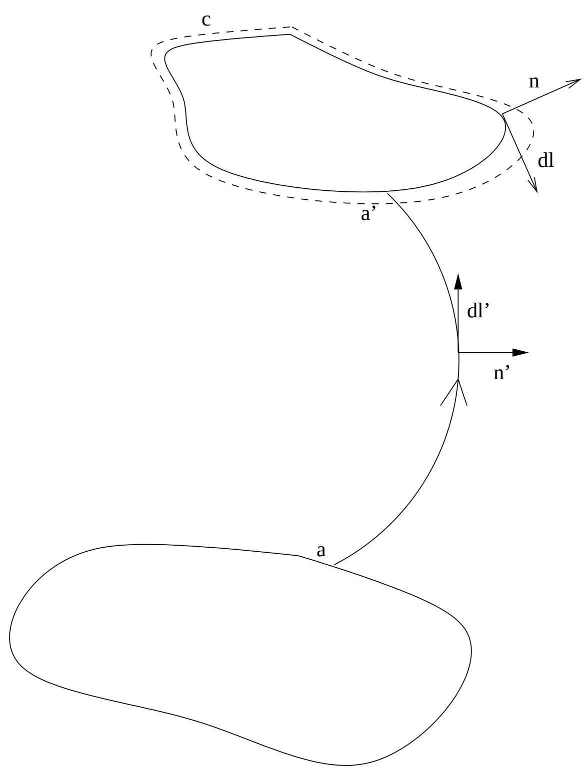

Figure 4:

Now, the inductance per unit length is given by

$$l = \lim_{\Delta z \to 0} \frac{F_{\Delta z}}{I} = \frac{\mu}{I} \lim_{\Delta z \to 0} \int \int \vec{B} \cdot d\vec{a_s} = \frac{\mu}{I} \int_a^{a'} \vec{B} \cdot \vec{n'} dl' = -\mu \frac{\int_a^{a'} \vec{B} \cdot \vec{n'} dl'}{\oint_c \vec{B} \cdot \vec{dl}}, \quad (32)$$

whereas the capacitance per unit length is given by

$$c = \frac{Q_{\Delta z}^{enclosed}}{V} = \frac{\epsilon}{V} \lim_{\Delta z \to 0} \int \int_s E \cdot d\vec{a_s} = \frac{\epsilon}{V} \oint_c \vec{E} \cdot \hat{n} dl = \epsilon \frac{\oint_c \vec{E} \cdot \hat{n} dl}{-\int_a^{a'} \vec{E} \cdot \vec{dl'}}, \quad (33)$$

where we have considered the a segment of the transmission line of length Δz, and let $\Delta z \to 0$.

So, we find that

$$cl = -\mu \frac{\int_a^{a'} \vec{B} \cdot \vec{n'} dl'}{\oint_c \vec{B} \cdot \vec{dl}} \epsilon \frac{\oint_c \vec{E} \cdot \hat{n} dl}{-\int_a^{a'} \vec{E} \cdot \vec{dl'}} \quad (34)$$

$$= \mu\epsilon \frac{\int_a^{a'} (\hat{z} \times \vec{E}) \cdot \vec{n'} dl'}{\oint_c (\hat{z} \times \vec{E}) \cdot \vec{dl}} \frac{\oint_c \vec{E} \cdot \hat{n} dl}{\int_a^{a'} \vec{E} \cdot \vec{dl'}} \quad (35)$$

$$= \mu\epsilon \frac{\int_a^{a'} \vec{E} \cdot \vec{dl'}}{\oint_c \vec{E} \cdot \hat{n} dl} \frac{\oint_c \vec{E} \cdot \hat{n} dl}{\int_a^{a'} \vec{E} \cdot \vec{dl'}} \quad (36)$$

$$= \mu\epsilon \quad (37)$$

5.33

(a)

The mutual inductance is given by

$$M_{12}(\vec{R}) = \frac{\mu_0}{4\pi} \oint \oint \frac{\vec{dl_1} \cdot \vec{dl_2}}{|\vec{x_1} - \vec{x_2} + \vec{R}|}, \quad (38)$$

which implies

$$\vec{\bigtriangledown}_R M_{12}(\vec{R}) = \frac{\mu_0}{4\pi} \oint \oint \vec{\bigtriangledown}_R \left(\frac{1}{|\vec{x_1} - \vec{x_2} + \vec{R}|}\right) \vec{dl_1} \cdot \vec{dl_2} \quad (39)$$

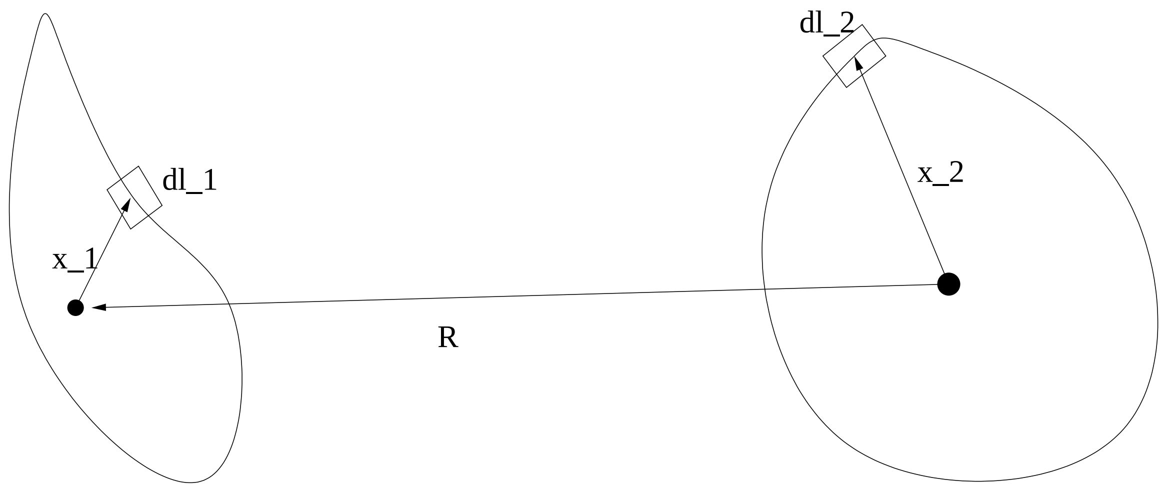

Figure 5:

$$= -\frac{\mu_0}{4\pi} \oint \oint (\frac{\vec{x_1} - \vec{x_2} + \vec{R}}{|\vec{x_1} - \vec{x_2} + \vec{R}|^3}) \vec{dl_1} \cdot \vec{dl_2} = -\frac{\mu_0}{4\pi} \oint \oint (\frac{\vec{x_{12}}}{|\vec{x_{12}}|^3}) \vec{dl_1} \cdot \vec{dl_2}. \qquad (40)$$

Combining this with Eq. (5.10),

$$\vec{F_{12}} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{(\vec{dl_1} \cdot \vec{dl_2}) \vec{x_{12}}}{|\vec{x_{12}}|^3}, \qquad (41)$$

we find

$$\vec{F_{12}} = I_1 I_2 \vec{\nabla}_R M_{12}(\vec{R}). \qquad (42)$$

(b)

Well,

$$\nabla_R^2 M_{12}(\vec{R}) = \frac{\mu_0}{4\pi} \oint \oint \nabla_R^2 (\frac{1}{|\vec{x_1} - \vec{x_2} + \vec{R}|}) \vec{dl_1} \cdot \vec{dl_2} \qquad (43)$$

$$= \frac{\mu_0}{4\pi} \oint \oint (-4\pi \delta(\vec{x_1} - \vec{x_2} + \vec{R})) \vec{dl_1} \cdot \vec{dl_2} = \mu_0 \oint \oint \delta(\vec{x_1} - \vec{x_2} + \vec{R}) \vec{dl_1} \cdot \vec{dl_2}. \quad (44)$$

We restrict to the situation $\vec{x_1} - \vec{x_2} + \vec{R} \neq 0$ – we do not want our loops to touch each other. In this case, $\delta(\vec{x_1} - \vec{x_2} + \vec{R}) = 0$ over the range of integrations, so we find

$$\nabla_R^2 M_{12}(\vec{R}) = 0. \tag{45}$$

6.1

a)The retarded solution is

$$\psi^+(\vec{x},t) = \int \frac{\delta(x')\delta(y')\delta(t - \frac{|\vec{x}-\vec{x'}|}{c})}{|\vec{x}-\vec{x'}|} dx'dy'dz'. \tag{46}$$

Doing the x' and y' integrations, we find

$$\psi^+(\vec{x},t) = \int \frac{\delta(t - \frac{\sqrt{x^2+y^2+(z-z')^2}}{c})}{\sqrt{x^2+y^2+(z-z')^2}} dz'. \tag{47}$$

We can now take advantage of the fact that $\int \delta(f(z'))g(z)dz' = \sum_{z_0} \frac{g(z_0)}{|f'(z_0)|}$, where z_0 are the values of z for which $f(z) = 0$. Now, for $\sqrt{x^2+y^2} > ct$ the argument of the delta function is always less than zero, so that we have

$$\psi(\vec{x}, t < \frac{\sqrt{x^2+y^2}}{c}) = 0. \tag{48}$$

For $\sqrt{x^2+y^2} < ct$ the argument in the delta function has two zeros, and the above formula allows us to find

$$\psi(\vec{x}, t > \frac{\sqrt{x^2+y^2}}{c}) = \frac{c}{\sqrt{c^2t^2 - x^2 - y^2}}. \tag{49}$$

Combining these, we get the result

$$\psi(\vec{x},t) = \Theta(ct - \sqrt{x^2+y^2}) \frac{2c}{\sqrt{c^2t^2 - x^2 - y^2}}. \tag{50}$$

b)

We can use the result of (a) to help solve this question. In particular,

$$\psi^+(\vec{x},t) = \int \frac{\delta(x')\delta(t - \frac{|\vec{x}-\vec{x'}|}{c})}{|\vec{x}-\vec{x'}|} dx'dy'dz' \tag{51}$$

$$= \int \Big(\int \frac{\delta(t - \frac{\sqrt{x^2+(y-y')^2+(z-z')^2}}{c})}{\sqrt{x^2+(y-y')^2+(z-z')^2}} dz' \Big) dy' \tag{52}$$

$$= \int \Theta(ct - \sqrt{x^2+(y-y')^2}) \frac{2c}{\sqrt{c^2t^2 - x^2 - (y-y')^2}} dy'. \tag{53}$$

Now, if $ct < |x|$, for all y', the theta function factor is zero. So,

$$\psi^+(\vec{x}, t < \frac{|x|}{c}) = 0. \tag{54}$$

For $ct > |x|$,

$$\psi^+(\vec{x}, t > \frac{|x|}{c}) = \int \Theta(ct - \sqrt{x^2+(y-y')^2}) \frac{2c}{\sqrt{c^2t^2 - x^2 - (y-y')^2}} dy' \tag{55}$$

$$= \int_{y-\sqrt{c^2t^2-x^2}}^{y+\sqrt{c^2t^2-x^2}} \frac{2c}{\sqrt{c^2t^2 - x^2 - (y-y')^2}} dy' \tag{56}$$

$$= \Big[2c \sin^{-1}(\frac{y'-y}{\sqrt{c^2t^2-x^2}}) \Big]_{y-\sqrt{c^2t^2-x^2}}^{y+\sqrt{c^2t^2-x^2}} = 2\pi c \tag{57}$$

Combining these two regimes gives

$$\psi^+(\vec{x}, t) = 2\pi c \Theta(ct - |x|). \tag{58}$$

Appendix

We will show that for $s > 1$

$$\int_0^{2\pi} \log(s - \cos\phi) d\phi = 2\pi \log(\frac{1}{2}(s + \sqrt{s^2 - 1})). \tag{59}$$

To begin with, define

$$I(s) = \int_0^{2\pi} \log(s - \cos\phi) d\phi \tag{60}$$

Notice that it is also true that

$$I(s) = \int_0^{2\pi} \log(s + \cos\phi) d\phi. \tag{61}$$

Then,

$$I'(s) = \int_0^{2\pi} \frac{1}{s + \cos\phi} d\phi = \int_\gamma \frac{1}{s + (1/2)(z + 1/z)} \frac{dz}{iz}, \tag{62}$$

where γ is the unit circle in the complex plane.
Now

$$\int_\gamma \frac{1}{s + (1/2)(z + 1/z)} \frac{dz}{iz} = \int_\gamma \frac{-2i}{z^2 + 2sz + 1} dz \tag{63}$$

$$= \int_\gamma \frac{-2i}{(z + s + \sqrt{s^2 - 1})(z + d - \sqrt{s^2 - 1})} dz \tag{64}$$

$$= \frac{2\pi}{\sqrt{s^2 - 1}}, \tag{65}$$

where we have noted that $\frac{-2i}{(z+s+\sqrt{s^2-1})(z+s-\sqrt{s^2-1})}$ has one residue inside γ at $z = \sqrt{s^2 - 1} - s$.
Now,

$$I'(s) = \frac{2\pi}{\sqrt{s^2 - 1}}, \tag{66}$$

so

$$I(s) = 2\pi \log(s + \sqrt{s^2 - 1}) + C. \tag{67}$$

In the limit of large s, we must have

$$I(s) \to 2\pi \log(s), \tag{68}$$

which allows us to identify the constant as $-2\pi \log 2$. Thus,

$$I(s) = 2\pi \log(s + \sqrt{s^2 - 1}) - 2\pi \log 2 = 2\pi \log(\frac{1}{2}(s + \sqrt{s^2 - 1})) \tag{69}$$