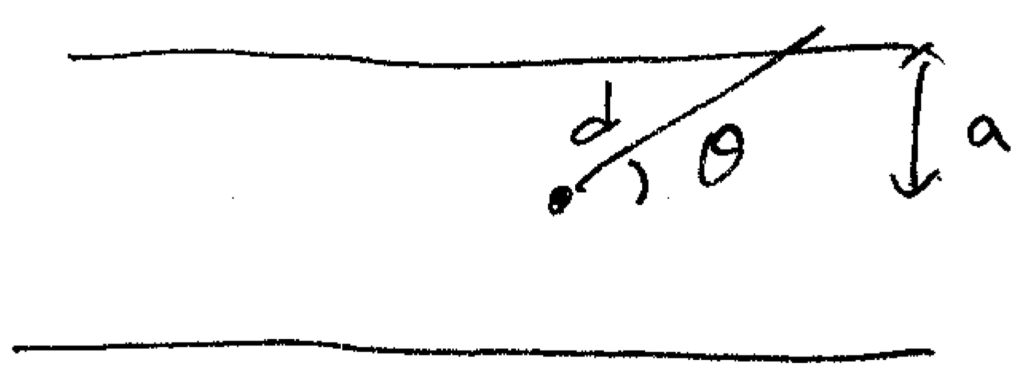
5-3



from Biot and Savart

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{\ell} \times \hat{r}}{r^2}$$

for one loop

$$B_z = \frac{\mu_0}{4z} \frac{I\, 2\pi a \sin\theta}{d^2} = \frac{\mu_0}{4\pi} \frac{I\, 2\pi \sin^3\theta}{a}$$

$\uparrow$
$a = d\sin\theta$

As $NL \to \infty$, $dN = N\,dz$ and $\frac{d\theta}{dz} = \frac{\sin\theta}{d}$ so $dN = N\frac{a\,d\theta}{\sin^2\theta}$

$$B_{z,\,total} = \int B_z\, dN = \frac{\mu_0}{4\pi} I\, 2\pi N \int_{\theta_2}^{\pi-\theta_1} \sin\theta\, d\theta = \boxed{\frac{\mu_0 I N}{2}(\cos\theta_1 + \cos\theta_2)}$$

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5-10 from 5.33

(a) we have $J_\phi = I\,\delta(z)\,\delta(\rho - a)$ in cylindrical coords.

we take the observation point x on the x ais, so its coordinates are $(\rho, \phi=0, z)$. Since there is no current in the z direction, and since the current density is cylindrically symmetric, there is no vector potential in the ρ or z direction. In the ϕ direction

$$A_\phi = -A_x \sin\phi + A_y \cos\phi = A_y$$

$$= \frac{\mu_0}{4\pi}\int \frac{J_y(x')}{|x-x'|}dx' = \frac{\mu_0}{4\pi}\int \frac{J_\phi(x')\cos\phi'}{|x-x'|}dx' = \frac{\mu_0}{4\pi}\mathrm{Re}\int \frac{J_\phi(x')e^{i\phi'}}{|x-x'|}dx'$$

$$= \frac{\mu_0}{4\pi}\mathrm{Re}\int J_\phi(x')e^{i\phi'}\left[\frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty e^{im(\phi-\phi')}\cos[k(z-z')\, I_m(k\rho_<)\, K_m(k\rho_>)\, dk\right]dx'$$

$\uparrow$
3.148

$$A_\phi = \frac{\mu_0}{2\pi^2}\mathrm{Re}\sum_{m=-\infty}^{\infty}\int_0^\infty \left[\int J_\phi(x')\, e^{i(1-m)\phi'}\cos[k(z-z')\, I_m(k\rho_<)\, K_m(k\rho_>)\, dx'\right]dk.$$

over $\rightarrow$

If $m=1$, ϕ integral yields 2π. otherwise zero

$$A_\phi = \frac{\mu_0}{\pi}\int_0^\infty \left[\int_0^\infty\int_{-\infty}^\infty J_\phi(r',z')\cos[k(z-z')]\, I_1(k\rho_<)\, K_1(k\rho_>)\,\rho' dz' dr'\right] dk$$

using $J_\phi = I\,\delta(z)\,\delta(\rho-a)$

$$\boxed{A_\phi = \frac{I a \mu_0}{\pi}\int_0^\infty \cos(kz)\, I_1(k\rho_<)\, K_1(k\rho_>)\, dk}$$

(b) from problem 3.16 (b)

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') e^{-k|z|}$$

Note $z'=0$, $\phi=0$ so substituting it into 5.35

$$A_\phi(r,\theta) = \frac{\mu_0 I}{4\pi a}\int r'^2 dr' d\Omega' \sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im\phi'} J_m(k\rho) J_m(k\rho') e^{-k|z|} \sin\theta' \cos\phi' \delta(\cos\theta')\delta(r'-a)$$

only $m=1$ survives then

$$\boxed{A_\phi = \frac{\mu_0 I a}{2}\int_0^\infty dk\, e^{-k|z|} J_1(ka) J_1(k\rho)}$$

(c) suppose the observation point is in the interior region, so $\rho_< = \rho$, $\rho_> = a$, then

$$\boxed{B_\rho} = (\nabla\times A)_\rho = -\frac{\partial A_\phi}{\partial z} = \boxed{-\frac{I a \mu_0}{\pi}\int_0^\infty k \sin kz\, I_1(k\rho)\, K_1(ka)\, dk}$$

$$\boxed{B_z} = (\nabla\times A)_z = \frac{1}{\rho}A_\phi + \frac{\partial A_\phi}{\partial \rho} = \boxed{\frac{I a \mu_0}{\pi}\int_0^\infty \cos kz \left[\frac{I_1(k\rho)}{\rho} + k I_1'(k\rho)\right] K_1(ka)\, dk}$$

As $\rho=0$, $I_1(\rho)\to 0$, $I_1(\rho)/\rho \to \frac{1}{2}$, and $I_1'(\rho)\to\frac{1}{2}$, so

$$B_\rho(\rho=0) = 0$$

$$B_z(\rho=0) = \frac{I a \mu_0}{\pi}\int_0^\infty k \cos kz\, K_1(ka)\, dk$$

$$= \frac{I a \mu_0}{\pi}\frac{\partial}{\partial z}\int_0^\infty \sin kz\, K_1(ka)\, dk$$

integration by parts.

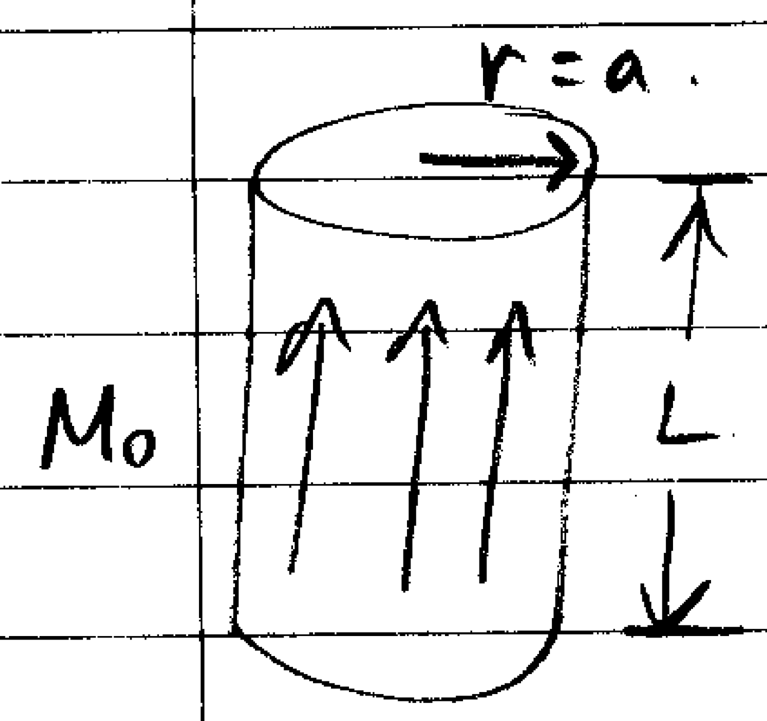
$$\int_0^\infty \sin kz \, K_1(kz)\, dk = \left| \cancelto{0}{-\frac{1}{a}\sin kz \, K_0(ka)} \right|_0^\infty + \frac{z}{a}\int_0^\infty \cos kz \, K_0(ka)\, dk$$

$$\left(\begin{array}{ll} K_0(0) \text{ finite} & \sin(0) = 0 \\ K_0(\infty) = 0 & \sin(\infty) : \text{finite} \end{array} \right)$$

for the second term , 3.150

$$B_z(\rho = 0) = \frac{I\mu_0}{2}\frac{\partial}{\partial z}\frac{1}{(z^2+a^2)^{\frac{1}{2}}} = \boxed{\frac{I\mu_0}{2}\frac{a^2}{(z^2+a^2)^{3/2}}}$$

Question 3: Jackson 5.19.



$$\vec{M} = M_0\,\hat{z}$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

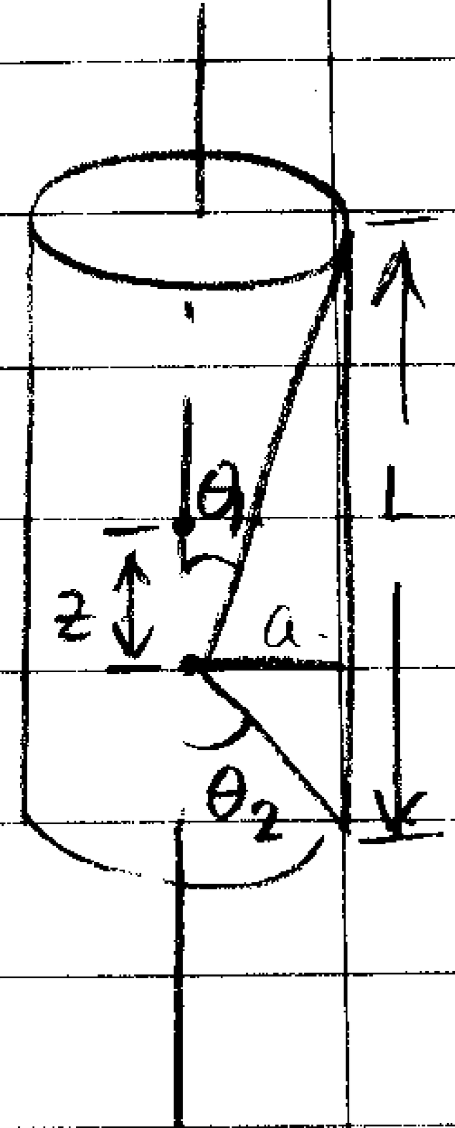
$$\vec{J} = \vec{\nabla}\times\vec{M} = 0 \qquad \text{current density.}$$

$$\vec{K} = \vec{M}\times\hat{n} \qquad \text{surface current}$$
$$= M_0\,\hat{z}\times\hat{\rho}$$
$$= M_0\,\hat{\varphi}$$

surface current flows on the material.
This is analogous to the case of the solenoid with $NI = M_0$ in Question 1: Jackson 5.3.

Taking the result from Q.1:

$$B_{z\,\text{total}} = \frac{\mu_0}{2} NI\left[\cos\theta_2 + \cos\theta_1\right]$$



$$B_z = \frac{\mu_0 M_0}{2}\left[\cos\theta_2 + \cos\theta_1\right]$$

take origin at center of cylinder

$$\cos\theta_1 = \frac{\frac{L}{2}+z}{\sqrt{a^2+\left(\frac{L}{2}+z\right)^2}}, \quad \cos\theta_2 = \frac{\frac{L}{2}-z}{\sqrt{a^2+\left(\frac{L}{2}-z\right)^2}}$$

$$\boxed{B_z = \frac{\mu_0 M_0}{2}\left[\frac{\frac{L}{2}+z}{\sqrt{a^2+\left(\frac{L}{2}+z\right)^2}} + \frac{\frac{L}{2}-z}{\sqrt{a^2+\left(\frac{L}{2}-z\right)^2}}\right]}$$

$\vec{B} = B_z\,\hat{z}$ which holds inside and outside of the cylinder.

$$\vec{H}_{in} = \frac{\vec{B}}{\mu_0} - \vec{M} \qquad \text{inside cylinder}, \ \vec{M} = M_0 \hat{z}$$

$$\boxed{H_{z\,in} = \frac{M_0}{2}\left[\frac{\frac{L}{2}+z}{\sqrt{a^2+(\frac{L}{2}+z)^2}} + \frac{\frac{L}{2}-z}{\sqrt{a^2+(\frac{L}{2}-z)^2}}\right] - M_0}$$

outside the cylinder, $\vec{M} = 0$.

$$\vec{H}_{out} = \vec{B}/\mu_0$$

$$\boxed{H_{z\,out} = \frac{M_0}{2}\left[\frac{\frac{L}{2}+z}{\sqrt{a^2+(\frac{L}{2}+z)^2}} + \frac{\frac{L}{2}-z}{\sqrt{a^2+(\frac{L}{2}-z)^2}}\right]}$$

b) plot $\frac{\vec{B}}{\mu_0 M_0}$ and $\frac{\vec{H}}{M_0}$ on the axis as fn. of z for $L/a = 5$.

$$\frac{B_z}{\mu_0 M_0} = \frac{1}{2}\left[\frac{\frac{L/a}{2} + z/a}{\sqrt{1+(\frac{1}{2}\frac{L}{a}+\frac{z}{a})^2}} + \frac{\frac{L}{2a} - \frac{z}{a}}{\sqrt{1+(\frac{1}{2}\frac{L}{a}-\frac{z}{a})^2}}\right]$$

$$\frac{B_z}{\mu_0 M_0} = \frac{1}{2}\left[\frac{\frac{5}{2} + z/a}{\sqrt{1+(\frac{5}{2}+\frac{z}{a})^2}} + \frac{\frac{5}{2} - z/a}{\sqrt{1+(\frac{5}{2}-z/a)^2}}\right] \quad \text{everywhere.}$$

$$\frac{H_z}{M_0} = \frac{B_z}{\mu_0 M_0} - 1 \qquad \text{for } -\frac{L}{2} < z < \frac{L}{2}$$

$$\frac{H_z}{M_0} = \frac{B_z}{\mu_0 M_0} \qquad \text{for } |z| > \frac{L}{2}.$$

$$B = 0.5 * \left(\frac{2.5 + z}{\sqrt{1 + (2.5 + z)^2}} + \frac{2.5 - z}{\sqrt{1 + (2.5 - z)^2}} \right);$$

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Plot[B, {z, -7, 7}]
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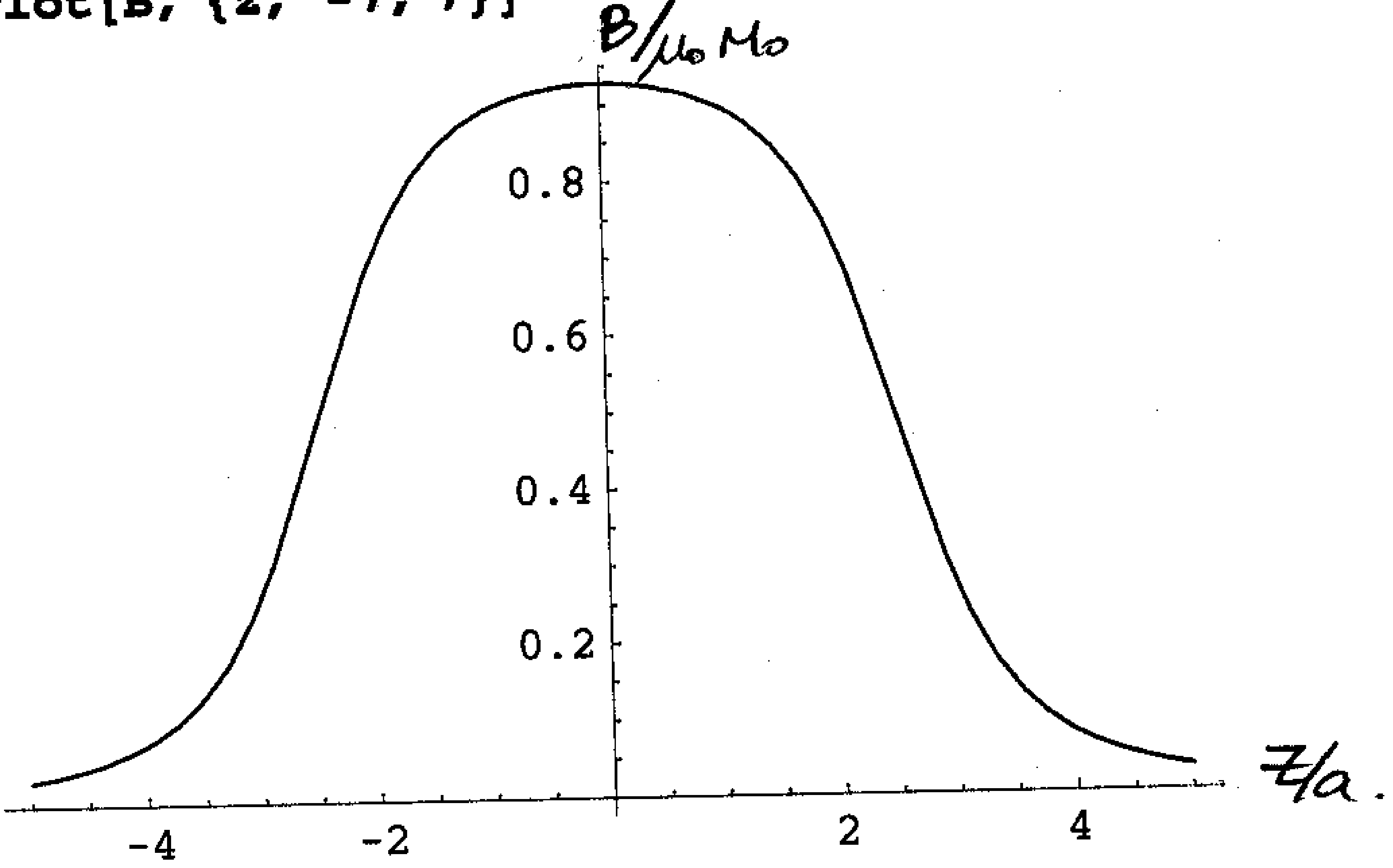

Out[18]= - Graphics -

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In[24]:= H = B - UnitStep[z + 2.5] + UnitStep[z - 2.5];
         Plot[H, {z, -7, 7}]
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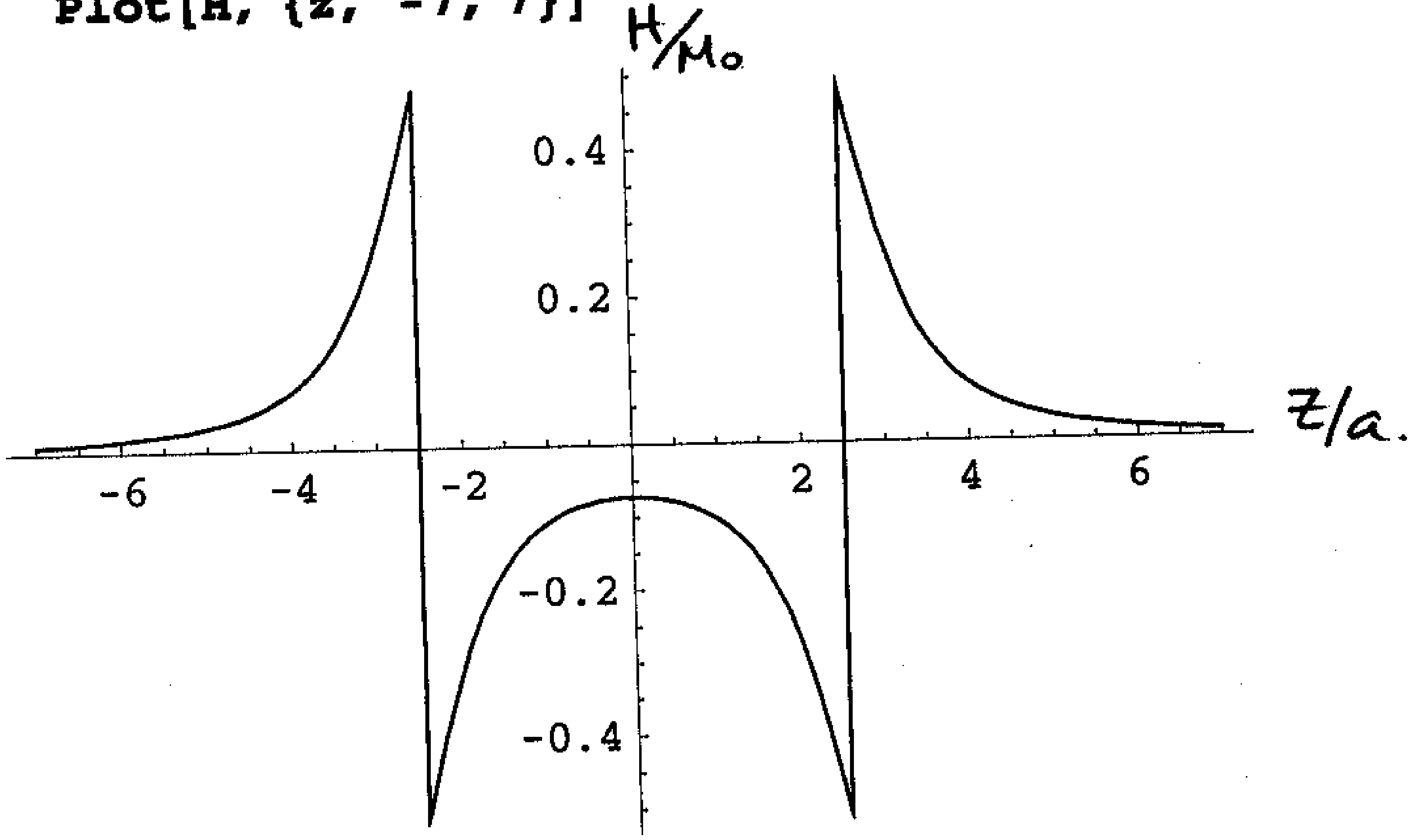

Out[25]= - Graphics -

5-21 (a) no free current

$$\vec{H} = -\nabla\Phi_M$$

$$\int_{\text{all space}} \vec{B}\cdot\vec{H}\,d^3x = -\int \vec{B}\cdot\nabla\Phi_M\,d^3x = -\int_{\substack{\text{all}\\ \text{space}}} [\vec{\nabla}\cdot(\Phi_M\vec{B}) - \Phi_M(\vec{\nabla}\cdot\vec{B})]\,d^3x$$

$$= -\int_{\substack{\text{all}\\ \text{space}}} \vec{\nabla}\cdot(\Phi_M\vec{B})\,d^3x = -\int_{S_\infty} \Phi_M(\vec{B}\cdot\hat{n})\,da$$

However, for $x\to\infty$, $\vec{B}\to 0$, so at S_∞ $\vec{B}=0$.

$$\therefore \boxed{\int_{\substack{\text{all}\\ \text{space}}} \vec{B}\cdot\vec{H}\,d^3x = 0}$$

(b) 5.72 $U = -\vec{m}\cdot\vec{B}$

Treating the distribution as a bunch of little dipoles,

$$W = -\frac{1}{2}\int \vec{M}\cdot\vec{B}\,d^3x, \quad \vec{B} = (\vec{H}+\vec{M})\mu_0$$

$\uparrow$ from double counting

$$W = -\frac{\mu_0}{2}\int \vec{M}\cdot(\vec{H}+\vec{M})\,d^3x = -\frac{\mu_0}{2}\int\vec{M}\cdot\vec{H}\,d^3x + \underbrace{\frac{\mu_0}{2}\int\vec{M}\cdot\vec{M}\,d^3x}_{\text{constant}}$$

within an additive constant

$$\boxed{W = -\frac{\mu_0}{2}\int \vec{M}\cdot\vec{H}\,d^3x}$$

using $\frac{\vec{B}}{\mu_0} = \vec{H}+\vec{M}$ again

$$W = -\frac{\mu_0}{2}\int\left(\frac{\vec{B}}{\mu_0}-\vec{H}\right)\cdot\vec{H}\,d^3x = \underbrace{-\frac{1}{2}\int\vec{B}\cdot\vec{H}\,d^3x}_{=0 \text{ from (a)}} + \frac{\mu_0}{2}\int\vec{H}\cdot\vec{H}\,d^3x$$

$$\therefore \boxed{W = \frac{\mu_0}{2}\int\vec{H}\cdot\vec{H}\,d^3x}$$

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