3.17 a) $\nabla^2 G(x,x') = -4\pi\,\delta(x-x') = -4\pi\,\frac{\delta(\rho-\rho')}{\rho'}\,\delta(\varphi-\varphi')\,\delta(z-z')$

Expand using completeness for $\delta(\varphi-\varphi')$ and $\delta(z-z')$:

$$\delta(\varphi-\varphi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-im\varphi'}e^{im\varphi} \qquad \delta(z-z') = \frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi z}{L}\sin\frac{n\pi z'}{L}$$

$$\Rightarrow \nabla^2 G(x,x') = -\frac{4}{L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{-im\varphi'}e^{im\varphi}\sin\frac{n\pi z}{L}\sin\frac{n\pi z'}{L}\,\frac{\delta(\rho-\rho')}{\rho'}$$

$$= \sum_{m,n}\left[-\frac{4}{L}e^{-im\varphi'}\sin\frac{n\pi z'}{L}\,\frac{\delta(\rho-\rho')}{\rho'}\right]e^{im\varphi}\sin\frac{n\pi z}{L} \qquad (1)$$

Expand G in same orthogonal function sets:

$$\nabla^2 G(x,x') = \nabla^2\sum_{m,n}A_{mn}(\rho|\rho',z',\varphi')\,e^{im\varphi}\sin\frac{n\pi z}{L}$$

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} \quad \text{so:}$$

$$\nabla^2 G(x,x') = \sum_{m,n}\left\{\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{m^2}{\rho^2} - \frac{n^2\pi^2}{L^2}\right]A_{mn}\right\}e^{im\varphi}\sin\frac{n\pi z}{L} \qquad (2)$$

Setting equal coefficients from (1) and (2), with: $A_{mn}(\rho|\rho',\varphi',z') = g_{mn}(\rho,\rho')\,e^{-im\varphi'}\sin\frac{n\pi z'}{L}$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g}{\partial\rho}\right) - \left(\frac{n^2\pi^2}{L^2} + \frac{m^2}{\rho^2}\right)g = -\frac{4}{L}\,\frac{\delta(\rho-\rho')}{\rho'} \qquad (3)$$

modified Bessel's Equation ($\rho \to i\rho$)

So general solution: $g_{mn}(\rho,\rho') = A\,I_m(k\rho) + B\,K_m(k\rho)$, $k = \frac{n\pi}{L}$, but we want $|g| < \infty$ and g continuous at $\rho = \rho'$ so $g_{mn}(\rho,\rho') = C\,I_m(k\rho_<)K_m(k\rho_>)$. Find C by integrating (3): $\int_{\rho'-\varepsilon}^{\rho'+\varepsilon} d\rho'$ and $\lim_{\varepsilon\to 0}$ gives:

$$\frac{d}{d\rho}g_{mn}(\rho,\rho')\Big|_{\rho'-\varepsilon}^{\rho'+\varepsilon} = -\frac{4}{\rho' L} = -Ck\cdot\frac{1}{k\rho'} \Rightarrow C = \frac{4}{L}$$

Plugging it all back in:

$$G(x,x') = \frac{4}{L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\varphi-\varphi')}\sin\frac{n\pi z}{L}\sin\frac{n\pi z'}{L}\,I_m\!\left(\frac{n\pi}{L}\rho_<\right)K_m\!\left(\frac{n\pi}{L}\rho_>\right).$$

b) Expand using completeness for $\frac{\delta(\rho-\rho')}{\rho'}$ and $\delta(\varphi-\varphi')$:

$$\frac{\delta(\rho-\rho')}{\rho'} = \int_0^\infty dk\, k J_m(k\rho') J_m(k\rho)$$

$$\Rightarrow \nabla^2 G(x,x') = \sum_{m=-\infty}^{\infty} \int_0^\infty dk \left[-2k e^{-im\varphi'} J_m(k\rho')\,\delta(z-z')\right] e^{im\varphi} J_m(k\rho)$$

$$= \nabla^2 \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, A_{mk}(z|\rho',\varphi',z')\, e^{im\varphi} J_m(k\rho) \qquad \cancel{\nabla^2} = \sum_{m=-\infty}^{\infty} \int_0^\infty dk \left\{ \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{m^2}{\rho^2} + \frac{\partial^2}{\partial z^2}\right] A_{mk} J_m(k\rho)\right\} \times e^{im\varphi}$$

~~Define: $A_{mk}(z,|\rho',\varphi',z') = a$~~

$$= \sum_{m=-\infty}^{\infty} \int_0^\infty dk \left\{\left[\frac{d^2}{dz^2} - k^2\right] A_{mk}\right\} e^{im\varphi} J_m(k\rho)$$

Define: $A_{mk}(z|\rho',\varphi',z') = a_k(z,z')\, e^{-im\varphi'} J_m(k\rho')$ and equating coeffs:

$$\frac{d^2 a}{dz^2} - k^2 a = -2k\,\delta(z-z') \quad \Rightarrow \quad g_k(z,z') = C \sinh(kz_<) \sinh(k(L-z_>))$$

Integrating to find C:

$$\frac{d}{dz} g_k(z,z')\Big|_{z'-\varepsilon}^{z'+\varepsilon} = -2k = -kC\sinh(kz')\cosh(k(L-z)) - kC\cosh(kz)\sinh(k(L-z'))$$

$$= -kC\sinh(kL) \quad \Rightarrow \quad C = \frac{2}{\sinh kL}$$

Then:

$$G(x,x') = 2\sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im(\varphi-\varphi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<)\sinh[k(L-z_>)]}{\sinh(kL)}$$

3.20 a) Center the z-coordinate axis on the point charge, so in our green's function from 3.17(a) only $K_m(\frac{n\pi\rho}{L})$ remains.

Charge distribution: $\rho(\vec{x}) = q\frac{\delta(\rho)}{2\pi\rho}\delta(z-z_0)$ or $\rho(\vec{x}') = q\frac{\delta(\rho')}{2\pi\rho'}\delta(z'-z_0)$

Then:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\,G(\vec{x},\vec{x}')\,d^3x'$$

$$= \frac{q}{\pi\varepsilon_0 L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\int_0^{\infty}d\rho'\int_0^{2\pi}\rho'd\varphi'\int_0^L dz\,\frac{\delta(\rho')}{2\pi\rho'}\delta(z'-z_0)\,e^{im(\varphi-\varphi')}\sin\frac{n\pi z}{L}\sin\frac{n\pi z'}{L}K_m\left(\frac{n\pi\rho}{L}\right)$$

$$= \frac{q}{\pi\varepsilon_0 L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\sin\frac{n\pi z}{L}\sin\frac{n\pi z_0}{L}K_m\left(\frac{n\pi\rho}{L}\right)\cdot\frac{1}{2\pi}\int_0^{2\pi}e^{im(\varphi-\varphi')}d\varphi'$$

Integral: $$\frac{1}{2\pi}e^{im\varphi}\int_0^{2\pi}e^{-im\varphi'}d\varphi' = \begin{cases}1, & m=0\\ \frac{1}{2\pi}e^{im\varphi}\left[-\frac{e^{-im\varphi'}}{im}\right|_0^{2\pi}, & m\neq 0\end{cases}$$

for $m\neq 0$: $$-\frac{1}{2\pi}e^{im\varphi}\cdot\frac{e^{-2\pi i m}-e^0}{im} = -\frac{1}{2\pi}e^{im\varphi}\cdot\frac{1-1}{im} = \underline{0}$$

So:

$$\Phi(\vec{x}) = \frac{q}{\pi\varepsilon_0 L}\sum_{n=1}^{\infty}\sin\frac{n\pi z}{L}\sin\frac{n\pi z_0}{L}K_0\left(\frac{n\pi\rho}{L}\right)$$

b) $$\sigma_0(\rho) = -\epsilon_0\frac{\partial\Phi}{\partial z}\Big|_{z=0} = -\frac{q}{\pi L}\sum_{n=1}^{\infty}\frac{n\pi}{L}\sin\frac{n\pi z_0}{L}K_0\left(\frac{n\pi\rho}{L}\right) = -\frac{q}{L^2}\sum_{n=1}^{\infty}n\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right)$$

$$\sigma_L(\rho) = +\epsilon_0\frac{\partial\Phi}{\partial z}\Big|_{z=L} = +\frac{q}{L^2}\sum_{n=1}^{\infty}n\cos(n\pi)\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right) = \frac{q}{L^2}\sum_{n=1}^{\infty}(-1)^n n\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right)$$

c) $$Q_L = \int_0^{\infty}\sigma_L(\rho)\,2\pi\rho\,d\rho = 2\pi q\cdot\sum_{n=1}^{\infty}(-1)^n n\sin\left(\frac{n\pi z_0}{L}\right)\int_0^{\infty}\frac{1}{n^2\pi^2}\frac{n\pi\rho}{L}K_0\left(\frac{n\pi\rho}{L}\right)\frac{n\pi\,d\rho}{L}$$

$$= \frac{2q}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi z_0}{L} = \frac{2q}{\pi}\,\mathrm{Im}\left[\sum_{n=1}^{\infty}\frac{(-e^{i\pi z_0/L})^n}{n}\right] = \frac{2q}{\pi}\,\mathrm{Im}\left[-\log\left(1+e^{i\pi z_0/L}\right)\right]$$

$$= \frac{2q}{\pi}\left(-\frac{\pi z_0}{2L}\right) = -q\frac{z_0}{L}$$

3.21

(a) The limit of the Green function

$$G(x,x') = 2\sum_m \int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<)\sinh(k(L-z_>))}{\sinh(kL)}$$

as $L \to \infty$ is

$$G(x,x') = 2\sum_m \int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \sinh(kz_<) e^{-kz_>}$$

If the charge density on the disk is given as $\sigma(\rho)$, the potential on the disk $(z_< = z_> = d)$ is

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} 2\pi \int_0^\infty dk \left(1-e^{-2kd}\right) J_0(k\rho) \int d\rho' \rho' \sigma(\rho') J_0(k\rho')$$

(the $m \neq 0$ terms vanish due to azimuthal symmetry)

For the <u>actual</u> $\sigma(\rho)$ corresponding to a conducting disk, $\Phi(x) = \text{const.}$ Therefore, we can write

$$\Phi = \frac{2\pi \int d\rho\, \rho\, \sigma(\rho)\, \Phi}{Q} = \frac{1}{4\pi\varepsilon_0} (2\pi)^2 \int_0^\infty dk \left(1-e^{-2kd}\right)\left[\int d\rho\, \rho\, \sigma(\rho) J_0(k\rho)\right]^2$$

$$\frac{4\pi\varepsilon_0}{C} = \frac{4\pi\varepsilon_0 \Phi}{Q} = \int_0^\infty dk \left(1-e^{-2kd}\right) \frac{\left[\int d\rho\, \rho\, \sigma(\rho) J_0(k\rho)\right]^2}{\left[\int d\rho\, \rho\, \sigma(\rho)\right]^2}$$

$$\left(Q = 2\pi \int d\rho\, \rho\, \sigma(\rho)\right)$$

b) assuming $\sigma(\rho) \to \sigma$

the

$$\frac{4\pi\varepsilon_0}{C} = \int_0^\infty dk(1-e^{-2kd})\, \frac{\cancel{\sigma^2}\left[\int_0^R \rho J_0(k\rho)d\rho\right]^2}{\cancel{\sigma^2}\left[\underbrace{\int_0^R \rho\, d\rho}_{R^2/2}\right]^2}$$

$$= 4\int_0^\infty dk(1-e^{-2kd})\left[\int_0^R \rho J_0(k\rho)\, d\rho\right]^2 / R^4$$

using $\int_0^R u J_0(u)du = R J_1(R) \to k^2\int_0^{R} \rho J_0(k\rho)d\rho = kR J_1(kR)$

$$\int_0^R \rho J_0(k\rho)d\rho = \frac{R}{k} J_1(kR)$$

then

$$\frac{4\pi\varepsilon_0}{C} = \int_0^\infty dk(1-e^{-2kd})\, 4\, \frac{\cancel{R^2}}{k^2} J_1(kR)^2 \frac{1}{\cancel{R^4}}$$

$$= \int_0^\infty dk(1-e^{-2kd})\, 4\left(\frac{J_1(kR)}{Rk}\right)^2 \qquad d \ll R \quad \text{taylor} \to \quad 1-(1-2kd) = 2kd$$

$$= \int_0^\infty dk\, 2kd\, 4\left(\frac{J_1(kR)}{Rk}\right)^2 = \int_0^\infty dk . \frac{8d}{R^2} . \frac{[J_1(kR)]^2}{k} = \frac{8d}{R^2}\underbrace{\int_0^\infty \frac{dk}{k} J_1(k)}_{1/2}$$

$$\frac{4\pi\varepsilon_0}{C} = \frac{4d}{R^2} \to \boxed{C = \frac{\pi\varepsilon_0 R^2}{d}}$$

now if $d \ll R$ we're approaching a parallel plate capacitor where $Q = \pi R^2 \sigma$

& as such $E = \sigma/\varepsilon_0$ thus $V = \frac{\sigma d}{\varepsilon_0}$ & $C = \frac{Q}{V} = \frac{\pi R^2 \cancel{\sigma}}{\cancel{\sigma} d/\varepsilon_0}$

$$\boxed{C = \frac{\pi R^2 \varepsilon_0}{d}}$$ indeed they are the same!

3.21

b) contd

using a previous result $\frac{4\pi\epsilon_0}{C} = \int_0^\infty dk(1-e^{-2kd})\, 4\left(\frac{J_1(kR)}{Rk}\right)^2$

if $d \gg R$ then $e^{-2kd} \approx 0$

thus

$$\frac{4\pi\epsilon_0}{C} = \int_0^\infty dk\, 4\left(\frac{J_1(kR)}{kR}\right)^2 = \frac{4}{R}\underbrace{\int_0^\infty \left(\frac{J_1(k)}{k}\right)^2 dk}_{\frac{4}{3\pi}}$$

$$\boxed{\frac{4\pi\epsilon_0}{C} = \frac{16}{3\pi R}}$$

actual

$$\frac{4\pi\epsilon_0}{C} = \frac{\pi}{2R}$$

ratio $\rightarrow \frac{16}{3\pi}\cdot\frac{2}{\pi} = \frac{32}{3\pi^2} = 1.081$ pretty good approximation!

4. Jackson 3.23

Point charge q is located at (ρ', ϕ', z') inside grounded $z=0$, $z=L$, $\rho=a$.

Potential inside box →

$$\Phi(x,x') = \frac{q}{4\pi\epsilon_0 a} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{e^{im(\phi-\phi')} J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{mn}\rho'}{a}\right)}{x_{mn} J_{m+1}^2(x_{mn}) \sinh\left(\frac{x_{mn}L}{a}\right)} \sinh\left[\frac{x_{mn}}{a} z_<\right] \times \sinh\left[\frac{x_{mn}}{a}(L-z_>)\right]$$

$$\Phi(x) = \frac{q}{4\pi\epsilon_0} G(x,x')$$

Try $G(x,x') = \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \sum_{n=1}^{\infty} J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{mn}\rho'}{a}\right) g_{mn}(z,z')$

Need $g_{mn} \sim \sinh\left(\frac{x_{mn} z_<}{a}\right) \sinh\left(\frac{x_{mn}}{a}(L-z_>)\right)$

$$\nabla^2 G(x,x') = -4\pi\,\delta(x-x')$$

$$= -4\pi\,\delta(z-z') \sum \frac{e^{im(\phi-\phi')}}{2\pi} \sum \frac{J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{mn}\rho'}{a}\right)}{\frac{a^2}{2}[J_{m+1}(x_{mn})]^2}$$

since $\frac{\delta(\rho-\rho')}{\rho} = \frac{2}{a^2} \sum \frac{J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{mn}\rho'}{a}\right)}{[J_{m+1}(x_{mn})]^2}$

so $\sum\sum e^{im(\phi-\phi')} J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{mn}\rho'}{a}\right) \left[-\left(\frac{x_{mn}}{a}\right)^2 g_{mn} + \frac{d^2 g_{mn}}{dz^2}\right]$

$$= -4\pi\,\delta(z-z') \sum \frac{e^{im(\phi-\phi')}}{2\pi} \sum \frac{J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{mn}\rho'}{a}\right)}{\frac{a^2}{2}[J_{m+1}(x_{mn})]^2}$$

$$\frac{d^2 g_{mn}}{dz^2} - \left(\frac{x_{mn}}{a}\right)^2 g_{mn} = -\frac{4}{a^2} \frac{\delta(z-z')}{[J_{m+1}(x_{mn})]^2}$$

so $g_{mn} = C_{mn} \sinh\left(\frac{x_{mn} z}{a}\right) \sinh\left(\frac{x_{mn}}{a}(z'-L)\right) \quad 0 \le z \le z'$

$= C_{mn} \sinh\left(\frac{x_{mn} z'}{a}\right) \sinh\left(\frac{x_{mn}}{a}(z-L)\right) \quad z' \le z \le L$

$\lim_{\epsilon\to 0} \int_{z'-\epsilon}^{z'+\epsilon} dz$ (I)

$\therefore \frac{\partial g_{mn}}{\partial z} = C_{mn} \frac{x_{mn}}{a} \cosh\left(x_{mn}\frac{z}{a}\right) \sinh\left(\frac{x_{mn}}{a}(z'-L)\right)$

$= C_{mn} \frac{x_{mn}}{a} \sinh\left(x_{mn}\frac{z'}{a}\right) \cosh\left(\frac{x_{mn}}{a}(z-L)\right)$

$$\frac{x_{mn}}{a} C_{mn} \left\{ \sinh\left(\frac{x_{mn} z'}{a}\right) \cosh\left(\frac{x_{mn}}{a}(z'-L)\right) - \cosh\left(x_{mn}\frac{z'}{a}\right) \sinh\left(x_{mn}(z'-L)\right) \right\}$$

$$= -\frac{4}{a^2} \frac{1}{[J_{m+1}(x_{mn})]^2}$$

$\Longrightarrow \sinh\left[\frac{x_{mn}}{a}(z'-z'+L)\right] = \sinh\left(\frac{x_{mn}L}{a}\right)$

$$\Rightarrow C_{mn} = \frac{4}{x_{mn} a \sinh\left(x_{mn}\frac{L}{a}\right) [J_{m+1}(x_{mn})]^2}$$

So $\Phi(x) = \frac{q}{\pi\epsilon_0 a} \sum\sum e^{im(\phi-\phi')} \left(\frac{J_m(x_{mn}\rho/a) J_m(x_{mn}\rho'/a)}{x_{mn}[J_{m+1}(x_{mn})]^2}\right) \left(\frac{\sinh[x_{mn} z_</a] \sinh\ldots}{\sinh(x_{mn}\frac{L}{a})}\right)$

Next, $\Phi(x,x') = \frac{2}{\pi\epsilon_0 L} \sum_{m=-\infty}^{\infty} \sum_{n=1} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) \frac{I_m\left(\frac{n\pi\rho_<}{L}\right)}{I_m\left(\frac{n\pi a}{L}\right)}$

$\rightsquigarrow G(x,x') = \sum e^{im(\phi-\phi')} \sum \sin\frac{n\pi z}{L} \sin\frac{n\pi z'}{L} g_{mn}(\rho,\rho')$

$$g_{mn} \sim I_m\left(\frac{n\pi}{L}\rho_<\right) K_m\left(\frac{n\pi}{L}\rho_>\right)$$

Same as before...

$$\sum\sum e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)\left\{\frac{d^2 g_{mn}}{d\rho^2} + \frac{1}{\rho}\frac{dg_{mn}}{d\rho} - \left(\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m}{\rho}\right)^2\right) g_{mn}\right\}$$

$$= -4\pi\delta(x-x') = \frac{-4\pi\delta(\rho-\rho')}{\rho} \sum \frac{e^{im(\phi-\phi')}}{2\pi} \sum \frac{\sin\frac{n\pi z}{L}\sin\frac{n\pi z'}{L}}{(L/2)}$$

So long as $\rho \neq \rho'$ $\quad \frac{d^2 g_{mn}}{d\rho^2} + \frac{1}{\rho}\frac{dg_{mn}}{d\rho} - \left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m}{\rho}\right)^2\right] g_{mn} = -\frac{4}{L}\frac{\delta(\rho-\rho')}{\rho}$ ✱

has solns $I_m\left(\frac{n\pi\rho}{L}\right)$, $K_m\left(\frac{n\pi\rho}{L}\right)$

$$\text{so} \quad g_{mn} = \begin{cases} a_{mn} I_m\left(\frac{n\pi}{L}\rho\right) & 0 \le \rho \le \rho' \\ b_{mn} I_m\left(\frac{n\pi\rho}{L}\right) + c_{mn} K_m\left(\frac{n\pi\rho}{L}\right) & \rho' \le \rho \le a \end{cases}$$

→ $g_{mn} = 0$ at $\rho = a$

so $\quad 0 = b_{mn} I_m\left(\frac{n\pi a}{L}\right) + c_{mn} K_m\left(\frac{n\pi a}{L}\right)$

→ continuous at $\rho = \rho'$

$$a_{mn} I_m\left(\frac{n\pi\rho'}{L}\right) = b_{mn} I_m\left(\frac{n\pi}{L}\rho'\right) + c_{mn}\left(\frac{n\pi}{L}\rho'\right)$$

so $\quad c_{mn} = -b_{mn} I_m\left(\frac{n\pi a}{L}\right) / K_m\left(\frac{n\pi a}{L}\right)$

$\quad a_{mn} = b_{mn}\left[1 - \frac{I_m\left(\frac{n\pi a}{L}\right) K_m\left(\frac{n\pi\rho'}{L}\right)}{I_m\left(\frac{n\pi\rho'}{L}\right) K_m\left(\frac{n\pi a}{L}\right)}\right]$

So NOW
$$g_{mn} = \begin{cases} b_{mn}\left[1 - \frac{I_m\left(\frac{n\pi a}{L}\right) K_m\left(\frac{n\pi\rho'}{L}\right)}{I_m\left(\frac{n\pi\rho'}{L}\right) K_m\left(\frac{n\pi a}{L}\right)}\right] I_m\left(\frac{n\pi\rho}{L}\right) & \rho \le \rho' \\ b_{mn}\left[I_m\left(\frac{n\pi\rho}{L}\right) - \frac{I_m\left(\frac{n\pi a}{L}\right) K_m\left(\frac{n\pi\rho}{L}\right)}{K_m\left(\frac{n\pi a}{L}\right)}\right] & \rho \geq \rho' \end{cases}$$

$$\frac{dg_{mn}}{d\rho} = \begin{cases} b_{mn}\left[1 - \frac{I_m\left(\frac{n\pi a}{L}\right) K_m\left(\frac{n\pi\rho'}{L}\right)}{I_m\left(\frac{n\pi\rho'}{L}\right) K_m\left(\frac{n\pi a}{L}\right)}\right] \frac{d}{d\rho} I_m\left(\frac{n\pi\rho}{L}\right) \\ b_{mn}\left[\frac{d}{d\rho} I_m\left(\frac{n\pi\rho}{L}\right) - \frac{I_m\left(\frac{n\pi a}{L}\right)}{K_m\left(\frac{n\pi a}{L}\right)} \frac{d}{d\rho} K_m\left(\frac{n\pi\rho}{L}\right)\right] \end{cases}$$

$$\lim_{\varepsilon\to 0}\int_{\rho'-\varepsilon}^{\rho'+\varepsilon} ✱\, d\rho = \lim_{\varepsilon\to 0}\left\{\left.\frac{dg_{mn}}{d\rho}\right|_{\rho'+\varepsilon} - \left.\frac{dg_{mn}}{d\rho}\right|_{\rho'-\varepsilon}\right\} = -\frac{4}{L\rho'}$$

$$= b_{mn}\left\{\cancel{\left.\frac{dI_m\left(\frac{n\pi\rho}{L}\right)}{d\rho}\right|_{\rho'}} - \frac{I_m\left(\frac{n\pi a}{L}\right)}{K_m\left(\frac{n\pi a}{L}\right)} \left.\frac{dK_m\left(\frac{n\pi\rho}{L}\right)}{d\rho}\right|_{\rho'}\right.$$

$$\left. - \cancel{\left.\frac{dI_m\left(\frac{n\pi\rho}{L}\right)}{d\rho}\right|_{\rho'}} + \frac{I_m\left(\frac{n\pi a}{L}\right)}{K_m\left(\frac{n\pi a}{L}\right)} \frac{K_m\left(\frac{n\pi\rho'}{L}\right)}{I_m\left(\frac{n\pi\rho'}{L}\right)} \frac{d}{d\rho} I_m\left(\frac{n\pi\rho}{L}\right)\right|_{\rho'}$$

4...

$$b_{mn}\frac{I_m(\frac{n\pi a}{L})}{K_m(\frac{n\pi a}{L})}\left\{\frac{K_m(\frac{n\pi\rho'}{L})\frac{dI_m}{d\rho}\Big|_{\rho'}-I_m(\frac{n\pi\rho'}{L})\frac{dK_m}{d\rho}\Big|_{\rho'}}{I_m(\frac{n\pi\rho'}{L})}\right\}=-\frac{4}{L\rho'}$$

$$\text{numerator}=\frac{n\pi}{L}\left(K_m(\tfrac{n\pi\rho'}{L})\,I_m'(\tfrac{n\pi\rho'}{L})-I_m(\tfrac{n\pi\rho'}{L})\,K_m'(\tfrac{n\pi\rho'}{L})\right)$$

$$=\frac{n\pi}{L}W\left[K_m(x),I_m(x)\right]$$

$$=\cancel{\frac{n\pi}{L}}\;\frac{\cancel{L}}{\cancel{n\pi}\rho'}$$

$$b_{mn}\frac{I_m(\frac{n\pi a}{L})}{K_m(\frac{n\pi a}{L})}\cdot\frac{1}{\rho' I_m(\frac{n\pi\rho'}{L})}=-\frac{4}{L\rho'}$$

$$\text{so } b_{mn}=-\frac{4}{L}\frac{K_m(\frac{n\pi a}{L})}{I_m(\frac{n\pi a}{L})}I_m\left(\frac{n\pi\rho'}{L}\right)$$

$$\text{so } g_{mn}=\begin{cases}-\frac{4}{L}\left[\frac{K_m(\frac{n\pi a}{L})}{I_m(\frac{n\pi a}{L})}I_m\left(\frac{n\pi\rho'}{L}\right)-K_m\left(\frac{n\pi\rho'}{L}\right)\right]I_m\left(\frac{n\pi\rho}{L}\right) & \rho\le\rho'\\ -\frac{4}{L}\left[\frac{K_m(\frac{n\pi a}{L})}{I_m(\frac{n\pi a}{L})}I_m\left(\frac{n\pi\rho}{L}\right)-K_m\left(\frac{n\pi\rho}{L}\right)\right]I_m\left(\frac{n\pi\rho'}{L}\right) & \rho\ge\rho'\end{cases}$$

$$g_{mn}(\rho',\rho)=-\frac{4}{L}\left[\frac{K_m(\frac{n\pi a}{L})}{I_m(\frac{n\pi a}{L})}I_m\left(\frac{n\pi\rho_>}{L}\right)-K_m\left(\frac{n\pi\rho_>}{L}\right)\right]I_m\left(\frac{n\pi\rho_<}{L}\right)$$

$$\text{so } \Phi=\frac{q}{4\pi\varepsilon_0 L}\sum\sum e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)\frac{I_m\left(\frac{n\pi\rho_<}{L}\right)}{I_m\left(\frac{n\pi a}{L}\right)}$$

$$\times\left[K_m\left(\frac{n\pi a}{L}\right)I_m\left(\frac{n\pi\rho_>}{L}\right)-I_m\left(\frac{n\pi a}{L}\right)K_m\left(\frac{n\pi\rho_>}{L}\right)\right]$$

✓ check.

Last one...

$$\Phi(x,x') = \frac{2q}{\pi\epsilon_0 L a^2}\sum_{m=-\infty}^{\infty}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{e^{im(\phi-\phi')}\sin\left(\frac{k\pi z}{L}\right)\sin\left(\frac{k\pi z'}{L}\right) J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{mn}\rho'}{a}\right)}{\left[\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2\right] J_{m+1}^2(x_{mn})}$$

$$G(x,x') = \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sum_{k=1}^{\infty}\sin\frac{k\pi z}{L}\sin\frac{k\pi z'}{L}\sum_{n=1}^{\infty} A_{mkn} J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(x_{mn}\frac{\rho'}{a}\right)$$

solns $\rightarrow$ $\psi_{mkn} = \left(\frac{1}{\sqrt{2\pi}} e^{im\phi}\right)\left(\sqrt{\frac{2}{L}}\sin\left(\frac{k\pi z}{L}\right)\right)\left(\frac{J_m(x_{mn}\frac{\rho}{a})}{\frac{a}{\sqrt{2}} J_{m+1}(x_{mn})}\right)$

$\psi_{mkn} = 0$ on S

$$G(x,x') = 4\pi\sum_{mkn}\frac{\psi^*_{mkn}\psi_{mkn}}{k_{mkn}^2} \quad ✱$$

$$\nabla^2\psi_{mkn} = \frac{d^2\psi}{d\rho^2}+\frac{1}{\rho}\frac{d\psi}{d\rho}+\frac{1}{\rho^2}\frac{d^2\psi}{d\phi^2}+\frac{d^2\psi}{dz^2} = \frac{d^2\psi}{d\rho^2}+\frac{1}{\rho}\frac{d\psi}{d\rho}-\frac{m^2}{\rho^2}\psi-\frac{k^2\pi^2}{L^2}\psi$$

$$\frac{d^2\psi}{d\rho^2}+\frac{1}{\rho}\frac{d\psi}{d\rho} = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{L}}\frac{e^{im\phi}\sin\left(\frac{k\pi z}{L}\right)}{\frac{a}{\sqrt{2}}J_{m+1}(x_{mn})}\left\{\underbrace{\frac{\partial^2 J_m(x_{mn}\frac{\rho}{a})}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial J_m(x_{mn}\frac{\rho}{a})}{\partial\rho}}_{=\left[\frac{m^2}{\rho^2}-\left(\frac{x_{mn}}{a}\right)^2\right]J_m\left(x_{mn}\frac{\rho}{a}\right)}\right\}$$

$$\text{So } \nabla^2\psi_{kmn} = \left[\frac{m^2}{\rho^2}-\left(\frac{x_{mn}}{a}\right)^2\right]\psi_{kmn}-\frac{m^2}{\rho^2}\psi_{mkn}-\frac{k^2\pi^2}{L^2}\psi_{mkn}$$

$$= -\left[\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2\right]\psi_{mkn} = -k^2_{mkn}\psi_{mkn}$$

$$\text{so } k_{mn}^2 = \left(\frac{x_{mn}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2$$

$$✱ * \frac{q}{4\pi\epsilon_0} \Rightarrow \Phi = \frac{2q}{\pi\epsilon_0 a^2 L}\sum\sum\sum e^{im(\phi-\phi')}\sin\frac{k\pi z}{L}\sin\frac{k\pi z'}{L}\left\{\frac{J_m(x_{mn}\frac{\rho}{a})J(x_{mn}\frac{\rho'}{a})}{\left[\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2\right]J_{m+1}^2}\right\}$$

✓ check.