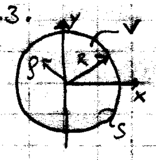
HW# 3 | Ph 106 - E+M | Erin Koos

Jackson 3.3, 3.5, 3.6, #4

3.3.



$\sigma = C(R^2-\rho^2)^{-1/2} \Rightarrow \Phi(\vec{x} \text{ on z axis}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\vec{x}')\,da'}{|\vec{x}-\vec{x}'|}$

a. $r > R$

$\Phi(|z|=r) = \frac{1}{4\pi\varepsilon_0}\int_0^R \frac{\sigma(\rho)(2\pi\rho)\,d\rho}{\sqrt{\rho^2+r^2}} = \frac{C}{2\varepsilon_0}\int_0^R \frac{\rho\,d\rho}{\sqrt{\rho^2+r^2}\sqrt{R^2-\rho^2}}$ $\quad u^2 = R^2-\rho^2,\ \rho^2 = R^2-u^2,\ 2u\,du = -2\rho\,d\rho$

$= \frac{C}{2\varepsilon_0}\int_0^R \frac{du}{\sqrt{r^2+R^2-u^2}} = \frac{C}{2\varepsilon_0}\left[\sin^{-1}\frac{u}{\sqrt{r^2+R^2}}\right]_0^R = \frac{C}{2\varepsilon_0}\sin^{-1}\frac{R}{\sqrt{r^2+R^2}}$

on disk, $\Phi(z=0) = V = \frac{C}{2\varepsilon_0}\sin^{-1}(1) = \frac{C}{2\varepsilon_0}\frac{\pi}{2} \Rightarrow C = \frac{4\varepsilon_0}{\pi}V$

$\therefore \Phi(|z|=r) = \frac{2V}{\pi}\sin^{-1}\frac{R}{\sqrt{R^2+r^2}} = \frac{2V}{\pi}\tan^{-1}\frac{R}{r}$ $\qquad \tan^{-1}x = \begin{cases} x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots & x<1 \rightarrow \frac{R}{r}<1 \\ \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \dots & x>1 \end{cases}$

$= \frac{2V}{\pi}\left[\frac{R}{r} - \frac{1}{3}\left(\frac{R}{r}\right)^3 + \frac{1}{5}\left(\frac{R}{r}\right)^5 - \dots\right] = \frac{2V}{\pi}\frac{R}{r}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left(\frac{R}{r}\right)^{2k} P_{2k}(\cos\theta)$

$$\boxed{\Phi(r,\theta,\phi) = \frac{2V}{\pi}\frac{R}{r}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{2\ell+1}\left(\frac{R}{r}\right)^{2\ell} P_{2\ell}(\cos\theta)} \quad r > R$$

b. $\Phi(|z|=r) = \frac{2V}{\pi}\tan^{-1}\frac{R}{r}$ $\qquad \tan^{-1}x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} \dots \quad \frac{R}{r}>1$

$= \frac{2V}{\pi}\left[\frac{\pi}{2} - \frac{r}{R} + \frac{1}{3}\left(\frac{r}{R}\right)^3 - \frac{1}{5}\left(\frac{r}{R}\right)^5 + \dots\right] = V - \frac{2V}{\pi}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{2\ell+1}\left(\frac{r}{R}\right)^{2\ell+1}$

$$\boxed{\Phi(r,\theta,\phi) = V - \frac{2V}{\pi}\frac{r}{R}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{2\ell+1}\left(\frac{r}{R}\right)^{2\ell} P_{2\ell+1}(|\cos\theta|)}, \quad r < R$$

c. $Q = \int_S \sigma\,ds = \frac{4\varepsilon_0 V}{\pi}\int_0^R \frac{2\pi\rho\,d\rho}{\sqrt{R^2-\rho^2}} = 8\varepsilon_0 V\left[-\sqrt{R^2-\rho^2}\right]_0^R = 8\varepsilon_0 VR$

$C = \frac{Q}{V} \Rightarrow \boxed{C = 8\varepsilon_0 R}$

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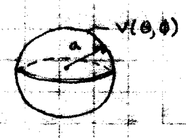
3.5.



a. $G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{x'|\vec{x} - \frac{a^2}{x'^2}\vec{x}'|}$

$= \frac{1}{(r^2+r'^2-2rr'\cos(\theta-\theta'))^{1/2}} - \frac{1}{\left(\frac{r^2r'^2}{a^2}+a^2-2rr'\cos(\theta-\theta')\right)^{1/2}} \quad \gamma: \theta-\theta'$

$\left.\frac{\partial G}{\partial r'}\right|_{r'=a} = \frac{(r^2-a^2)}{(r^2+a^2-2ar\cos(\gamma))}$

$$\therefore \boxed{\Phi(r,\theta,\phi) = \frac{1}{4\pi}\oint V(\theta',\phi')\frac{a(r^2-a^2)}{(r^2+a^2-2ar\cos\gamma)^{3/2}}\,d\Omega}$$

b. the general solution: $\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[C_{\ell m}r^\ell + D_{\ell m}r^{-(\ell+1)}\right]Y_{\ell m}(\theta,\phi)$

Φ finite at $r=0 \Rightarrow D_{\ell m} = 0$

$\Phi(a,\theta,\phi) = V(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} C_{\ell m}a^\ell Y_{\ell m}(\theta,\phi)$

$\int V(\theta,\phi)Y^*_{\ell'm'}(\theta,\phi)\,d\Omega = \sum_{\ell,m} C_{\ell m}a^\ell\delta_{\ell\ell'}\delta_{mm'} = C_{\ell'm'}a^{\ell}$

$C_{\ell m} = \frac{1}{a^\ell}\int V(\theta,\phi)Y^*_{\ell m}(\theta,\phi)\,d\Omega$

$$\therefore \boxed{\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}A_{\ell m}\left(\frac{r}{a}\right)^\ell Y_{\ell m}(\theta,\phi), \quad A_{\ell m} = \int d\Omega'\,Y^*_{\ell m}(\theta',\phi')V(\theta',\phi')}$$

cont.

3.5 cont

prove that these are the same –

$$\Phi(r,\theta,\phi) = \int V(\theta',\phi')\,d\Omega' \sum_{\ell=0}^{\infty} \left(\frac{r}{a}\right)^{\ell} \frac{2\ell+1}{4\pi} \underbrace{\frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi)}_{P_\ell(\cos\gamma)}$$

$$= \frac{1}{4\pi}\int V(\theta',\phi')\,d\Omega' \left[2\sum_{\ell=0}^{\infty} \ell \left(\frac{r}{a}\right)^{\ell} P_\ell \cos(\gamma) + a \sum_{\ell=0}^{\infty} \frac{r^\ell}{a^{\ell+1}} P_\ell \cos(\gamma) \right]$$

$$= \frac{1}{4\pi}\int V(\theta',\phi')\,d\Omega' \left[2ar\frac{\partial}{\partial r} \sum_{\ell=0}^{\infty} \frac{r^\ell}{a^{\ell+1}} P_\ell \cos\gamma + \frac{a}{\sqrt{r^2+a^2-2ar\cos\gamma}} \right]$$

$$= \frac{a}{4\pi}\int V(\theta',\phi')\,d\Omega' \left[2r\frac{\partial}{\partial r} \frac{1}{\sqrt{r^2+a^2-2ar\cos\gamma}} + \frac{1}{\sqrt{r^2+a^2-2ar\cos\gamma}} \right]$$

$$= \frac{a}{4\pi}\int V(\theta',\phi')\,d\Omega \left[-r\frac{2r-2a\cos\gamma}{(r^2+a^2-2ar\cos\gamma)^{3/2}} + \frac{1}{\sqrt{r^2+a^2-2ar\cos\gamma}} \right]$$

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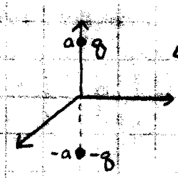
$$\boxed{\Phi(r,\theta,\phi) = \frac{a}{4\pi}(a^2-r^2)\int \frac{V(\theta',\phi')}{(r^2+a^2-2ar\cos\gamma)^{3/2}}\,d\Omega'}$$ as in part a.

3.6.



a. $\Phi(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{r}-a\hat{z}|} - \frac{1}{|\vec{r}+a\hat{z}|}\right]$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{r_>}\left[\left(1-2\left(\frac{r_<}{r_>}\right)\cos\gamma + \left(\frac{r_<}{r_>}\right)^2\right)^{-1/2} + \left(1+2\left(\frac{r_<}{r_>}\right)\cos\gamma + \left(\frac{r_<}{r_>}\right)^2\right)^{-1/2}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^{\ell}\left[P_\ell(\theta) - P_\ell(\pi-\theta)\right]$$

$$= \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^{\ell}\left(1+(-1)^{\ell+1}\right)P_\ell(\theta)$$

$$\boxed{\Phi(r,\theta,\phi) = \frac{q}{\sqrt{4\pi}\,\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^{\ell}\left(1+(-1)^{\ell+1}\right)Y_\ell^0(\theta,\phi)}$$

where $r_> = r$ if $r>a$ and $r_>$ a if $a>r$

$r_< = a$ if $r>a$ and $r_< = r$ if $a>r$

b. $qa = p/2 = \text{const}$ $\quad a\to 0 \quad r\neq 0$

$r>a$: $\Phi(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{r}\left(\frac{a}{r}\right)^{\ell}\left(1-(-1)^{\ell+1}\right)P_\ell(\theta)$

as $a\to 0$, $= \frac{2qa}{4\pi\epsilon_0 r^2}P_1(\theta) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$

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$\boxed{\Phi(r,\theta,\phi) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}}$, $a\to 0$

c. $\Phi_h = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\theta) \quad \Rightarrow \Phi(r,\theta,\phi) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} + \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\theta)$

$\Phi(r=b) = \frac{p\cos\theta}{4\pi\epsilon_0 b^2} + \sum_{\ell=0}^{\infty} A_\ell b^\ell P_\ell(\theta) = 0 \quad \Rightarrow A_\ell = \begin{cases} 0 & \ell\neq 1 \\ -\frac{p}{4\pi\epsilon_0 b^3}, & \ell=1 \end{cases}$

$$\boxed{\Phi(r,\theta,\phi) = \frac{p\cos\theta}{4\pi\epsilon_0}\left[\frac{1}{r^2} - \frac{r}{b^3}\right]}$$

(4) It is instructive to solve this problem in a more general context. Let us take the expression for the curl in Cartesian coordinates q_i

$$(\nabla \times A)_i = \varepsilon_{ike} \frac{\partial}{\partial q_k} A_e$$

and transform it to the general curvilinear coordinates $q_k'(q_i)$ with the metric

$$\gamma_{ik}' = \frac{\partial q_e}{\partial q_i'} \frac{\partial q_e}{\partial q_k'}$$

According to the vector transformation law,

$$(\nabla \times A)_i' = \frac{\partial q_i'}{\partial q_j} (\nabla \times A)_j = \frac{\partial q_i'}{\partial q_j} \varepsilon_{jke} \frac{\partial}{\partial q_k} A_e = \frac{\partial q_i'}{\partial q_j} \varepsilon_{jke} \frac{\partial q_s'}{\partial q_k} \frac{\partial}{\partial q_s'} A_e =$$

$$= \frac{\partial q_i'}{\partial q_j} \varepsilon_{jke} \frac{\partial q_s'}{\partial q_k} \frac{\partial q_t'}{\partial q_e} \frac{\partial q_m}{\partial q_t'} \frac{\partial}{\partial q_s'} A_m$$

where we've used $\frac{\partial q_t'}{\partial q_e} \frac{\partial q_m}{\partial q_t'} = \delta_{em}$. If we now take into account that

$$\varepsilon_{jke} \frac{\partial q_i'}{\partial q_j} \frac{\partial q_s'}{\partial q_k} \frac{\partial q_t'}{\partial q_e} = \det\left(\frac{\partial q_i'}{\partial q_j}\right) \varepsilon_{ist}$$

and $\det \gamma' = \left[\det\left(\frac{\partial q_e}{\partial q_i'}\right)\right]^2$, we obtain

$$(\nabla \times A)_i' = \frac{1}{\sqrt{\det \gamma'}} \varepsilon_{ist} \frac{\partial q_m}{\partial q_t'} \frac{\partial}{\partial q_s'} \left(\frac{\partial q_m}{\partial q_n'} A_n' \right) = \frac{1}{\sqrt{\det \gamma'}} \varepsilon_{ist} \frac{\partial}{\partial q_s'} \left(\frac{\partial q_m}{\partial q_t'} \frac{\partial q_m}{\partial q_n'} A_n' \right) =$$

$$= \frac{1}{\sqrt{\det \gamma'}} \varepsilon_{ike} \frac{\partial}{\partial q_k'} \left(\gamma_{em}' A_m' \right) \quad \left(\text{we've used the transformation law for } A_i \text{ and } \frac{\partial^2 q_m}{\partial q_s' \partial q_t'} = \frac{\partial^2 q_m}{\partial q_t' \partial q_s'} \right)$$

In other words, in arbitrary system of curvilinear coordinates

$$(\nabla \times A)_i = \frac{1}{\sqrt{\det \gamma}} \varepsilon_{ike} \frac{\partial}{\partial q_k} \left(\gamma_{em} A_m \right)$$

Substituting $A_i \equiv \vec{A} \cdot \vec{e}_i \equiv \vec{A} \cdot \hat{e}_i / h_i$ and $\gamma_{ik} = \mathrm{diag}(h_1^2, h_2^2, h_3^2)$ yields the desired result