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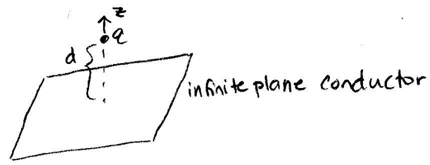
Physics 106b

Due 2-26-04

Homework #2 Jackson 2.1, 2.3, 2.7, 2.23

2.1



(a) Surface-charge density induced on the plane & plot.

from class notes, $\phi = \left[\frac{q}{|\vec{x} - d\hat{z}|} - \frac{q}{|\vec{x} + d\hat{z}|} \right] \cdot \frac{1}{4\pi\epsilon_0}$

$\rightarrow$ cylindrical coordinates (r, z, ϕ)

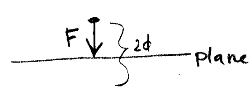
$$\phi = \left[\frac{q}{(r^2 + (z-d)^2)^{1/2}} - \frac{q}{(r^2 + (z+d)^2)^{1/2}} \right] \cdot \frac{1}{4\pi\epsilon_0}$$

$$\sigma = -\epsilon_0 \frac{\partial \phi}{\partial z}\bigg|_{z=0} = -\frac{q}{4\pi} \left[\frac{-\frac{1}{\cancel{2}}\cancel{(2)}(z-d)}{(r^2 + (z-d)^2)^{3/2}} - \frac{-\frac{1}{\cancel{2}}\cancel{(2)}(z+d)}{(r^2 + (z+d)^2)^{3/2}} \right] \bigg|_{z=0} = \frac{-qd}{2\pi(r^2+d^2)^{3/2}}$$

See plot at end of Pset...

(b) Force b/w plate and charge by Coulomb's law for force b/w charge & image

$$F = \frac{-q^2}{4\pi\epsilon_0 (2d)^2} = \frac{-q^2}{16\pi\epsilon_0 d^2}$$

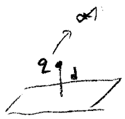


(c) Total force acting on plane $\rightarrow \int_{plane} \frac{\sigma^2}{2\epsilon_0}$

$$F = \int \frac{\sigma^2}{2\epsilon_0} da = \int \frac{q^2 d}{\epsilon_0 \cdot 8\pi^2 (r^2+d^2)^3} \cdot 2\pi r\, dr = \frac{q^2 d^2}{4\pi\epsilon_0} \int \frac{r\, dr}{(r^2+d^2)^3}$$

$$= \frac{q^2 d^2 (-\frac{1}{2} \cdot \frac{1}{2})}{4\pi\epsilon_0 (r^2+d^2)^2} \Bigg|_0^\infty = 0 - \frac{(-\frac{1}{4}) q^2 d^2}{4\pi\epsilon_0\, d^4} = \frac{q^2}{16\pi\epsilon_0 d^2}$$

(d) Work necessary to remove charge q from its position to infinity



$$W = -\int_d^{\infty} \vec{F}\cdot d\vec{l} = \int_d^{\infty} F_{c} dz = \frac{q^2}{16\pi\epsilon_0}\int_d^{\infty}\frac{dz}{z^2} = \frac{q^2}{16\pi\epsilon_0 d}$$

(e) Potential energy between charge q and its image

$$PE = W = \frac{1}{2}\cdot\frac{1}{4\pi\epsilon_0}\sum\frac{q_i q_j}{|\vec{x}_i-\vec{x}_j|} = -\frac{q^2}{8\pi\epsilon_0 d}$$

$\downarrow$

work to bring q from ∞ to d.

$W(d) + W(e) = -W_*$ $\quad$ $W_* \rightarrow$ work to create surface charge density σ.

(f) Part (d) in eV w $d = 1\,\text{Å}$

$$W = \frac{q^2}{16\pi\epsilon_0 d} = \frac{(1.6E\text{-}19\,C)^2}{4\,(1E\text{-}10\,m)(4\pi\epsilon_0)} = \frac{(1.6E\text{-}19\,C)^2}{4(1E\text{-}10\,m)}\left(\frac{9E9\,Nm^2}{C^2}\right)\left(\frac{eV}{1.6E\text{-}19\,J}\right)$$

$$= \boxed{3.6\,eV}$$

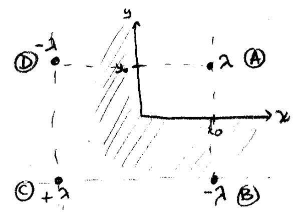
2.3 straight-line charge w/ constant linear charge density λ $\perp$ to x-y plane (x_0, y_0)

$\left.\begin{array}{l} x=0 \\ y \geq 0 \\ y=0 \\ x \geq 0 \end{array}\right\}$ conducting boundary surfaces held at zero potential

(a) $\phi(x,y) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{R^2}{r^2}\right)$ $\quad r^2 = (x-x_0)^2 + (y-y_0)^2$

R = constant

Determine potential of line charge in presence of planes
verify that the potential and tangential electric field vanish on boundary surfaces



$$r^2 = (x-x_0)^2 + (y-y_0)^2$$

$$\phi = \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(\frac{\cancel{R^2}}{r_A^2}\right) - \ln\left(\frac{\cancel{R^2}}{r_B^2}\right) + \ln\left(\frac{\cancel{R^2}}{r_C^2}\right) - \ln\left(\frac{\cancel{R^2}}{r_D^2}\right)\right]$$

$$= \frac{\lambda}{4\pi\epsilon_0} \ln \frac{r_B^2 r_D^2}{r_A^2 r_C^2} = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_B r_D}{r_A r_C}$$

$$r_A^2 = (x-x_0)^2 + (y-y_0)^2$$
$$r_B^2 = (x-x_0)^2 + (y+y_0)^2$$
$$r_C^2 = (x+x_0)^2 + (y+y_0)^2$$
$$r_D^2 = (x+x_0)^2 + (y-y_0)^2$$

$\left.\begin{array}{l} \text{when } x=0 \quad r_A = r_B,\ r_C = r_D \\ \text{when } y=0 \quad r_A = r_D,\ r_B = r_C \end{array}\right\}$ both cases $\ln(1) = 0$ so $\phi \to 0$ ✓

(b) surface charge density on plane $y=0$, $x \geq 0$ Plot σ/λ vs. x $(x_0=2, y_0=1)$ $(x_0=1, y_0=1)$ $(x_0=1, y_0=2)$

$$\sigma = -\epsilon_0 \frac{\partial\phi}{\partial y} = -\frac{\lambda}{2\pi}\left(\frac{1}{r_B}\frac{\partial r_B}{\partial y} + \frac{1}{r_D}\frac{\partial r_D}{\partial y} - \frac{1}{r_A}\frac{\partial r_A}{\partial y} - \frac{1}{r_C}\frac{\partial r_C}{\partial y}\right)\Big|_{y=0}$$

$$= -\frac{\lambda}{2\pi}\left(\frac{\cancel{y}+y_0}{r_B^2} + \frac{\cancel{y}-y_0}{r_D^2} - \frac{\cancel{y}-y_0}{r_A^2} - \frac{\cancel{y}+y_0}{r_C^2}\right)\Big|_{y=0}$$

$$= \frac{-\lambda y_0}{\pi}\left(\frac{1}{(x-x_0)^2 + y_0^2} - \frac{1}{(x+x_0)^2 + y_0^2}\right)$$

(c) Total charge (per unit length in z) on the plane $y=0$, $x \geq 0$

$$Q_x = -\frac{2}{\pi}\lambda \tan^{-1}\left(\frac{x_0}{y_0}\right) \qquad \text{Total charge on plane } x=0 \text{ ?}$$

$$Q_x = \int_0^\infty \sigma\, dx$$

$$= \frac{-y_0\lambda}{\pi}\left[\int_0^\infty \frac{dx}{(x-x_0)^2+y_0^2} - \int_0^\infty \frac{dx}{(x+x_0)^2+y_0^2}\right]$$

let $x - x_0 = y_0 \tan u$, $dx = y_0 \sec^2 u\, du$ — let $x + x_0 = y_0 \tan u$, $dx = y_0 \sec^2 u\, du$

$$= -\frac{\lambda}{\pi}\left[\int_{\tan^{-1}(-\frac{x_0}{y_0})}^{\pi/2} du - \int_{\tan^{-1}(\frac{x_0}{y_0})}^{\pi/2} du\right]$$

$$= -\frac{\lambda}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(-\frac{x_0}{y_0}\right) - \frac{\pi}{2} + \tan^{-1}\left(\frac{x_0}{y_0}\right)\right]$$

$$= -\frac{2\lambda}{\pi}\tan^{-1}\left(\frac{x_0}{y_0}\right) \quad \checkmark$$

total charge on plane $x=0$

$$Q_y = -\frac{2\lambda}{\pi}\tan^{-1}\left(\frac{y_0}{x_0}\right)$$ bc calculations symmetric w.r.t. x_0 and y_0.

(d) Far from the origin $\rho \gg \rho_0$ $\quad \rho = \sqrt{(x^2+y^2)}$, $\rho_0 = \sqrt{(x_0^2+y_0^2)}$

$$\phi \to \phi_{asym} = \frac{4\lambda}{\pi\epsilon_0}\frac{(x_0y_0)(xy)}{\rho^4}$$

$$\phi = \frac{\lambda}{4\pi\epsilon_0}\frac{r_B^2 r_D^2}{r_A^2 r_C^2}$$

$$r_A^2 = [(x-x_0)^2 + (y-y_0)^2] = x^2\left(1-\frac{x_0}{x}\right)^2 + y^2\left(1-\frac{y_0}{y}\right)^2$$

$$\approx x^2\left(1 - 2\frac{x_0}{x}\right) + y^2\left(1 - 2\frac{y_0}{y}\right)$$

$$= x^2 - 2x_0x + y^2 - 2y_0y$$

$$= (x^2+y^2)\left[1 - 2\frac{xx_0+yy_0}{x^2+y^2}\right]$$

so $r_B^2 = (x^2+y^2)\left[1 - \frac{2(-xx_0+yy_0)}{x^2+y^2}\right]$; $r_C^2 = (x^2+y^2)\left[1 - \frac{2(-xx_0-yy_0)}{x^2+y^2}\right]$; $r_D^2 = (x^2+y^2)\left[1 - 2\frac{xx_0-yy_0}{x^2+y^2}\right]$

$$r_A^2 r_C^2 = (x^2+y^2)^2\left[1 - \frac{4(xx_0+yy_0)^2}{(x^2+y^2)^2}\right] ; \quad r_B^2 r_D^2 = (x^2+y^2)\left[1 - 4\frac{(xx_0-yy_0)^2}{(x^2+y^2)^2}\right]$$

$$\phi = \frac{\lambda}{4\pi\epsilon_0}\left(\ln\left[\left(1 - \frac{4(xx_0-yy_0)^2}{(x^2+y^2)^2}\right)\right] - \ln\left[1 - \frac{4(xx_0+yy_0)^2}{(x^2+y^2)^2}\right]\right)$$

$$\ln(1+u) \approx u$$

$$\phi = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{-4(\cancel{x^2 x_0^2} + \cancel{y^2 y_0^2} - 2xy\, x_0 y_0) + 4(\cancel{x^2 x_0^2} + \cancel{y^2 y_0^2} + 2xy\, x_0 y_0)}{(x^2+y^2)^2}\right]$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[\frac{16\, xy\, x_0 y_0}{(x^2+y^2)^2}\right] = \frac{4\lambda\, xy\, x_0 y_0}{\pi\epsilon_0 (x^2+y^2)^2} \checkmark$$

2.7 Potential problem in half-space defined by $z \geqslant 0$

Dirichlet BCs in plane $z=0$ (and ∞)

$\hookrightarrow \phi$ known

(a) Green function $G(x,x')$

$$G(x,x') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{1}{|\vec{x}-\vec{x}''|}$$

$$\vec{x}' = x'\hat{\imath} + y'\hat{\jmath} + z'\hat{k}$$
$$\vec{x}'' = x'\hat{\imath} + y'\hat{\jmath} - z'\hat{k}$$

(b) potential on $z=0$ plane is $\phi = V$ inside of circle ($r=a$) centered at origin

$\phi = 0$ outside circle... Integral expression for potential at point P in terms of cylindrical coordinates (ρ, ϕ, z)

$$\phi(\vec{x}) = -\frac{1}{4\pi}\oint_S \phi(\vec{x}')\frac{\partial G}{\partial n'}da'$$

$\hat{n}'$ is in $-z$ direction

$$\phi(\vec{x}) = -\frac{1}{4\pi}\oint \phi(\vec{x}')\,da'\left(-\frac{\partial}{\partial z}G(\vec{x},\vec{x}')\Big|_{z'=0}\right) = \frac{-1}{4\pi}\oint\phi(\vec{x}')da'\left(\frac{-(z'-z)+(z'+z)}{(x-x')^2+(y-y')^2+z^2)}\right)$$

$\sim \frac{z}{z}$

$$= \quad '' \quad '' \left(-\frac{2z}{((x-x')^2+(y-y')^2+z^2)^{3/2}}\right)$$

$$\phi(\vec{x}) = \frac{z}{2\pi}V\int_0^a\int_0^{2\pi}\frac{\rho'd\rho'd\phi'}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}$$

$$x = \rho\cos\phi, \quad y = \rho\sin\phi,$$
$$x' = \rho'\cos\phi', \quad y' = \rho'\sin\phi'$$

$$(\rho\cos\phi' - \rho\cos\phi)(\rho\cos\phi - \rho\cos\phi')$$
$$\rho^2\cos^2\phi \, \cancel{-2\rho}\,\rho'\cos\phi\cos\phi' + \rho'^2\cos^2\phi'^2 \sim \text{same for } y \text{ w/ sins...}$$

$$\phi(\vec{x}) = z\cdot V\int_0^a\int_0^{2\pi}\frac{\rho'd\rho'd\phi'}{[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2]^{3/2}}$$

(c) along axis of circle ($\rho=0$) $\phi = V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right)$

$$\phi(\rho=0,\phi,z) = z\cdot V\cdot\int_0^a\frac{\rho'd\rho'}{(\rho'^2+z^2)^{3/2}} = z\cdot V\left[\frac{-1}{(\rho'^2+z^2)^{\frac{1}{2}}}\Big|_0^a\right] = V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right)$$

d) For $\rho^2+z^2 \gg a^2$ potential can be expanded in power series in $(\rho^2+z^2)^{-1}$

$$\Phi = \frac{Va^2}{2}\frac{z}{(\rho^2+z^2)^{3/2}}\left[1 - \frac{3a^2}{4(\rho^2+z^2)} + \frac{5(3\rho^2a^2+a^4)}{8(\rho^2+z^2)^2} + \cdots\right]$$

$$\Phi(\rho,\phi,z) = \frac{Vz}{2\pi}\int_0^{2\pi}\int_0^a \frac{d\phi'\rho'\,d\rho'}{(\rho^2+z^2)^{\frac{3}{2}}\left(1+\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right)^{\frac{3}{2}}}$$

binomial expansion $(1+x)^{\frac{3}{2}} = 1 - \frac{3}{2}x + \frac{15}{8}x^2 - \cdots$

$$1+\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2} = 1 - \frac{3}{2}\left[\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right] + \frac{15}{8}\left[\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right]^2$$

$$= 1 - \frac{3}{2}\left[\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right] + \frac{15}{8}\left[\frac{\rho'^4-4\rho\rho'^3\cos(\phi-\phi')+4\rho^2\rho'^2\cos^2(\phi-\phi')}{(\rho^2+z^2)^2}\right]$$

$$\Phi(\rho,\phi,z) = \frac{Vz}{2\pi(\rho^2+z^2)^{\frac{3}{2}}}\left[2\pi\int_0^a\rho'd\rho' - \frac{3}{2}\left[\frac{2\pi}{(\rho^2+z^2)}\right]\int_0^a\rho'^3d\rho' + \frac{15}{8}\left[\frac{2\pi}{(\rho^2+z^2)^2}\right]\cdot\right.$$

$$\left.\int_0^a\rho'^5d\rho' + 4\rho^2\pi\int_0^a\rho'^3d\rho'\right] \qquad \int_0^{2\pi}\cos(\phi-\phi')d\phi'=0 \qquad \int_0^{2\pi}\cos^2(\phi-\phi')d\phi'=\pi$$

$$\Phi(\rho,\phi,z) = \frac{Vz}{(\rho^2+z^2)^{\frac{3}{2}}}\left[\frac{a^2}{2} - \frac{3}{2(\rho^2+z^2)}\frac{a^4}{4} + \frac{15}{8}\frac{1}{(\rho^2+z^2)^2}\left[\frac{a^6}{6}+\frac{2\rho^2a^4}{4}\right] \cdots\right]$$

$$= \frac{Va^2z}{2(\rho^2+z^2)^{\frac{3}{2}}}\left[1 - \frac{3a^2}{4(\rho^2+z^2)} + \frac{5}{8}\frac{3\rho^2a^2+a^4}{(\rho^2+z^2)^2} + \cdots\right] \quad ✓✓$$

same as answer in (c) for $\rho=0$ and $\rho^2+z^2 \gg a^2$
$z^2 \gg a^2$

$$(d) \rightarrow \quad \Phi = \frac{Va^2}{2z^2}\left[1 - \frac{3a^2}{4z^2} + \frac{5a^4}{8z^4}\right]$$

$$(c) \rightarrow \quad \Phi = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right) = V\left(1 - \frac{z}{z\sqrt{1+\frac{a^2}{z^2}}}\right)$$

expand ...

$$\Phi = V\left(\cancel{1} - \cancel{\left[1\right.} - \frac{1}{2}\frac{a^2}{z^2} + \frac{3}{8}\left(\frac{a^2}{z^2}\right)^2 - \frac{5}{16}\left(\frac{a^2}{z^2}\right)^3 + \cdots\left.\right]\right)$$

$$= \frac{Va^2}{2z^2}\left[1 - \frac{3a^2}{4z^2} + \frac{5a^4}{8z^4}\right]$$

same!

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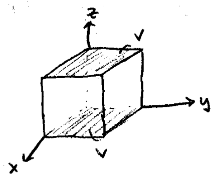
2.23



non-shaded areas are at zero potential

(a) $\phi(x,y,z)$ at any point inside cube

$$\nabla^2\phi = 0$$

$$\phi = F(x)\,G(y)\,H(z)$$

$$\rightarrow \frac{1}{F}\frac{\partial^2 F}{\partial x^2} + \frac{1}{G}\frac{\partial^2 G}{\partial y^2} + \frac{1}{H}\frac{\partial^2 H}{\partial z^2} = 0$$

$$\frac{1}{F}\frac{\partial^2 F}{\partial x^2} = -\alpha^2 \rightarrow F = \sin(\alpha_n x) \qquad \alpha_n = n\pi/a \quad n = 1,2,3\ldots$$

$$\frac{1}{G}\frac{\partial^2 G}{\partial y^2} = -\beta^2 \rightarrow G = \sin(\beta_m y) \qquad \beta_m = m\pi/a \quad m = 1,2,3\ldots$$

$$\gamma_{nm}^2 = \alpha_n^2 + \beta_m^2$$

$$\frac{1}{H}\frac{\partial^2 H}{\partial z^2} = \alpha^2 + \beta^2 = \gamma^2 \rightarrow H = \sinh(\gamma_{nm} z) + \cosh(\gamma_{nm} z)$$

so $$\phi = \sum_{n,m} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right)\left[A_{nm}\sinh(\gamma_{nm} z) + B_{nm}\cosh(\gamma_{nm} z)\right]$$

Boundary conditions, $z = 0, \phi = V$
$z = a, \phi = V$

$$\phi(x,y,0) = V = \sum_{n,m} B_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)$$

so $$B_{nm} = \frac{4V}{a^2}\int_0^a\int_0^a \underbrace{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)}\,dx\,dy$$

each integral same...
$= \frac{2a}{n\pi}$ for n odd
$= 0$ for n even

So $B_{nm} = \frac{16V}{nm\pi^2}$ for odd n, m

$$\phi(x,y,a) = V = \phi(x,y,0)$$

$$B_{nm} = A_{nm}\sinh(\gamma_{nm}a) + B_{nm}\cosh(\gamma_{nm}a)$$

so $$A_{nm} = \frac{B_{nm}\left(1-\cosh(\gamma_{nm}a)\right)}{\sinh(\gamma_{nm}a)}$$

so $$\phi(x,y,z) = \sum_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[\frac{16V\left(1-\cosh(\gamma_{nm}a)\right)}{nm\pi^2\sinh(\gamma_{nm}a)}\sinh(\gamma_{nm}z) + \frac{16V}{nm\pi^2}\cosh(\gamma_{nm}z)\right]$$

$\Rightarrow$ n, m odd!

(b) Potential at center of cube $(x,y,z) \to \left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$

$$\phi = \frac{16V}{\pi^2} \sum_{n,m} \frac{1}{nm} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{m\pi}{2}\right) \left[\frac{1-\cosh(\gamma_{nm} a)}{\sinh(\gamma_{nm} a)} \sinh\left(\frac{\gamma_{nm} a}{2}\right) + \cosh\left(\frac{\gamma_{nm} a}{2}\right) \right]$$

n, m odd (from navon!)

$$\frac{\gamma_{nm} a}{2} = \frac{a}{2}\sqrt{\alpha_n^2 + \beta_m^2} = \frac{\cancel{a}}{2}\sqrt{\left(\frac{n\pi}{\cancel{a}}\right)^2 + \left(\frac{m\pi}{\cancel{a}}\right)^2} = \frac{\gamma'}{2} \quad \text{where } \gamma'^2 = (n\pi)^2 + (m\pi)^2$$

$\boxed{n=1,\ m=1}$

$$\phi = \frac{16V}{\pi^2}\left[\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)\left[\frac{1-\cosh(\sqrt{2}\pi)}{\sinh(\sqrt{2}\pi)} \sinh\left(\frac{\pi}{\sqrt{2}}\right) + \cosh\left(\frac{\pi}{\sqrt{2}}\right)\right]\right] \approx V \cdot 0.3475$$

$\boxed{n=1,\ m=3}$

$$\phi = \frac{16V}{\pi^2}\left[\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{3\pi}{2}\right)\left[\frac{1-\cosh(\sqrt{10}\pi)}{\sinh(\sqrt{10}\pi)} \sinh\left(\frac{\sqrt{10}\,\pi}{2}\right) + \cosh\left(\frac{\sqrt{10}\pi}{2}\right)\right]\right] = -V 0.0075$$

$\boxed{n=3,\ m=1}$ same as above $\phi = -0.0075\,V$

adding these 3 gives $\phi \approx V \cdot 0.3325$

(c) Surface charge density on the surface $z=a$.

$$\sigma = -\epsilon_0 \frac{\partial \phi}{\partial z}\Big|_{z=a}$$

$$\frac{\partial \phi}{\partial z} = \frac{16V}{\pi^2} \sum_{n,m} \frac{1}{n \cdot m} \cdot \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \frac{\gamma'}{a}\left[\frac{1-\cosh(\gamma_{nm} a)}{\sinh(\gamma_{nm} a)} \cdot \cosh\left(\frac{\gamma' z}{a}\right) + \sinh\left(\frac{\gamma' z}{a}\right)\right]$$

$$-\epsilon_0 \frac{\partial \phi}{\partial z}\Big|_{z=a} = -\frac{\epsilon_0 \cdot 16V}{\pi^2} \sum_{n,m} \frac{\gamma'}{nma} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \left[\frac{1-\cosh(\gamma_{nm} a)}{\sinh(\gamma_{nm} a)} \cosh(\gamma') + \sinh \gamma' \right]$$