Linear Microwave Circuits:
From $\vec{\mathbf{a}}$ to Z_o

Outline

- Phasors
- Impedance
- Transmission Lines
- Wave Representation
- Scattering Matrices
- Smith Charts
- Johnson Noise
- Noise Wave Correlation Matrices
- Supermix **sdata** class
- Linear Connection Theory
- Subnet Growth in Supermix

Phasors

- Work in **frequency space**
- Sinusoids have **magnitude** and **phase**
- Complex number have magnitude and phase

We can represent V and I as complex numbers

$$V(\omega, t) = Re[V\sqrt{2}\, e^{j\omega t}]$$

$$I(\omega, t) = Re[I\sqrt{2}\, e^{j\omega t}]$$

Z may also be complex to show the relative phase between V and I

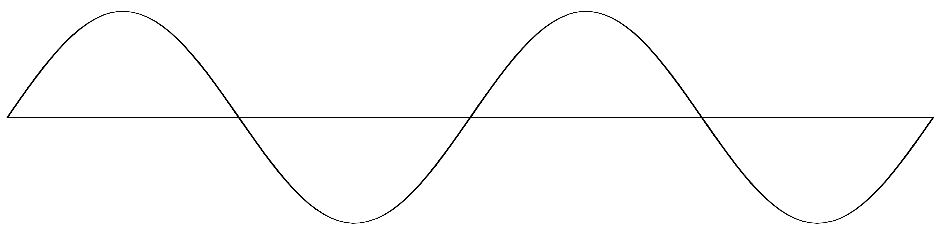
$$V = I\, Z$$

Capacitance

$$Q = CV$$

$$\frac{dQ}{dt} = C\,\frac{dV}{dt}$$

$$I = C\,\frac{dV}{dt}$$



$$V = V_o\, e^{j\omega t}$$

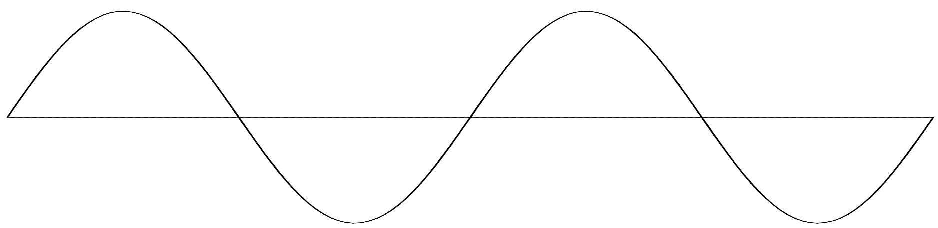
$$I = j\omega CV$$

$$Z = \frac{-j}{\omega C}$$

Current **LEADS** voltage by $\frac{\pi}{2}$

Inductance

$$V = L\,\frac{dI}{dt}$$



$$V = V_o\,e^{j\omega t}$$

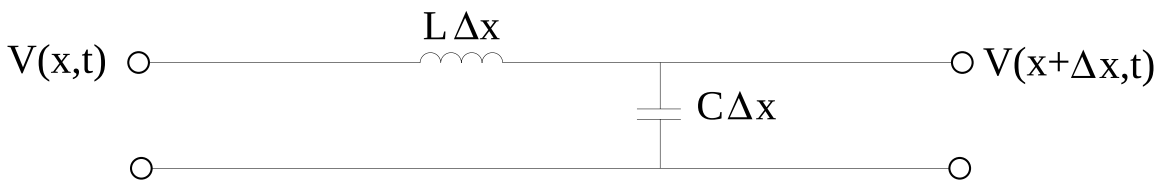
$$V_o\,e^{j\omega t}\,dt = L\,dI$$

$$\frac{1}{j\omega}\,V = LI$$

$$Z = j\omega L$$

Current **LAGS** voltage by $\frac{\pi}{2}$

Transmission Lines



Kirchhoff's Laws give

$$V(x+\Delta x,t) - V(x,t) = -L\,\Delta x\,\frac{\partial I}{\partial t}$$

$$I(x+\Delta x,t) - I(x,t) = -C\,\Delta x\,\frac{\partial V}{\partial t}$$

So

$$\frac{\partial V}{\partial x} = -L\,\frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial x} = -C\,\frac{\partial V}{\partial t}$$

This is equivalent to the wave equations:

$$\frac{\partial^2 V}{\partial x^2} = LC\,\frac{\partial^2 V}{\partial t^2}$$

$$\frac{\partial^2 I}{\partial x^2} = LC\,\frac{\partial^2 I}{\partial t^2}$$

Transmission Lines (Continued)

These give traveling wave solutions

$$V = V_o\, e^{j\omega(t-\sqrt{LC}\,x)}$$

$$I = I_o\, e^{j\omega(t-\sqrt{LC}\,x)}$$

The phase velocity is

$$c = \frac{1}{\sqrt{LC}}$$

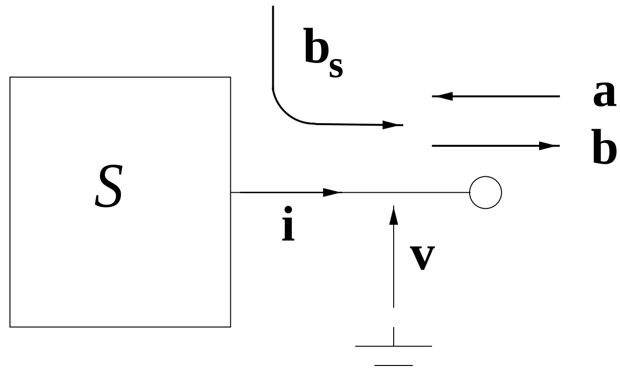
We can find Z from Kirchhoff's Laws

$$\frac{\partial I}{\partial x} = -C\,\frac{\partial V}{\partial t}$$

$$-j\omega\sqrt{LC}\,I = -j\omega C V$$

$$Z_o = \sqrt{\frac{L}{C}}$$

The Wave Representation



Incoming ($\mathbf{a}$) and outgoing ($\mathbf{b}$) waves are defined as

$$a_i = \frac{1}{2\sqrt{Z_0}}(V_i + Z_0 I_i) \quad \text{and} \quad b_i = \frac{1}{2\sqrt{Z_0}}(V_i - Z_0 I_i)$$

Power can be computed

$$P_i = |a_i|^2 - |b_i|^2 \quad \text{or} \quad P = \mathbf{a}^\dagger \mathbf{a} - \mathbf{b}^\dagger \mathbf{b}$$

The scattering matrix is defined by

$$\mathbf{b} = \mathcal{S}\mathbf{a} + \mathbf{b}_\mathrm{s}$$

$\mathcal{S}$ For Reciprocal Circuits

- Circuits are reciprocal if
  - Linear
  - Time invariant
  - Normal materials
  - No current or voltage sources
- Transmission is the same in both directions, so

$$S_{ij} = S_{ji}$$

- This tells us nothing about reflections (S_{11}, S_{22}, etc.)

$\mathcal{S}$ For Lossless Circuits

If a passive circuit has no loss, then

$$\mathbf{b}^\dagger \mathbf{b} = \mathbf{a}^\dagger \mathbf{a}$$

$$\mathbf{b} = \mathcal{S}\,\mathbf{a}$$

$$\mathbf{b}^\dagger = \mathbf{a}^\dagger\,\mathcal{S}^\dagger$$

$$\mathbf{a}^\dagger\,\mathcal{S}^\dagger\,\mathcal{S}\,\mathbf{a} = \mathbf{a}^\dagger \mathbf{a}$$

$$\mathcal{S}^\dagger\,\mathcal{S} = \mathcal{I}$$

$\Rightarrow$ $\mathcal{S}$ is unitary

Scattering Matrix Examples

- Ideal Transmission Line

$$\mathcal{S} = e^{-j\theta} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_{12} = S_{21} \quad \Rightarrow \quad \text{Reciprocal}$$

$$\mathcal{S}^{\dagger}\mathcal{S} = \mathcal{I} \quad \Rightarrow \quad \text{Lossless}$$

- Series Resistor

$$r \equiv \frac{R}{Z_o} \qquad \mathcal{S} = \frac{1}{2+r} \begin{pmatrix} r & 2 \\ 2 & r \end{pmatrix}$$

$$S_{12} = S_{21} \quad \Rightarrow \quad \text{Reciprocal}$$

$$\mathcal{S}^{\dagger}\mathcal{S} \neq \mathcal{I} \quad \Rightarrow \quad \text{Lossy}$$

- Amplifier

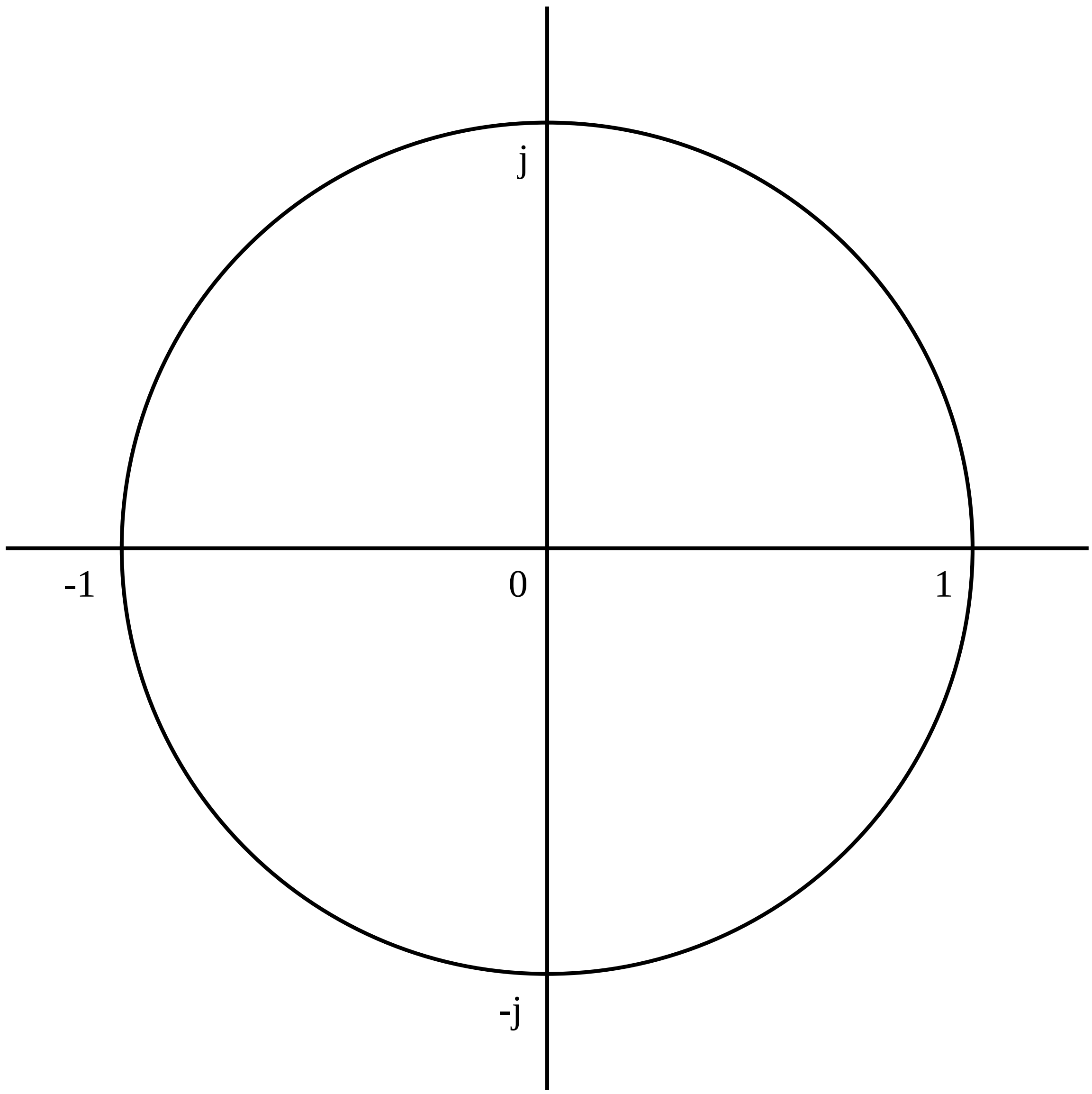
$$\mathcal{S} = \begin{pmatrix} 0 & 0 \\ 10 & 0 \end{pmatrix}$$

$$S_{12} \neq S_{21} \quad \Rightarrow \quad \text{Not Reciprocal}$$

$$\mathcal{S}^{\dagger}\mathcal{S} \neq \mathcal{I} \quad \Rightarrow \quad \text{Lossy}$$

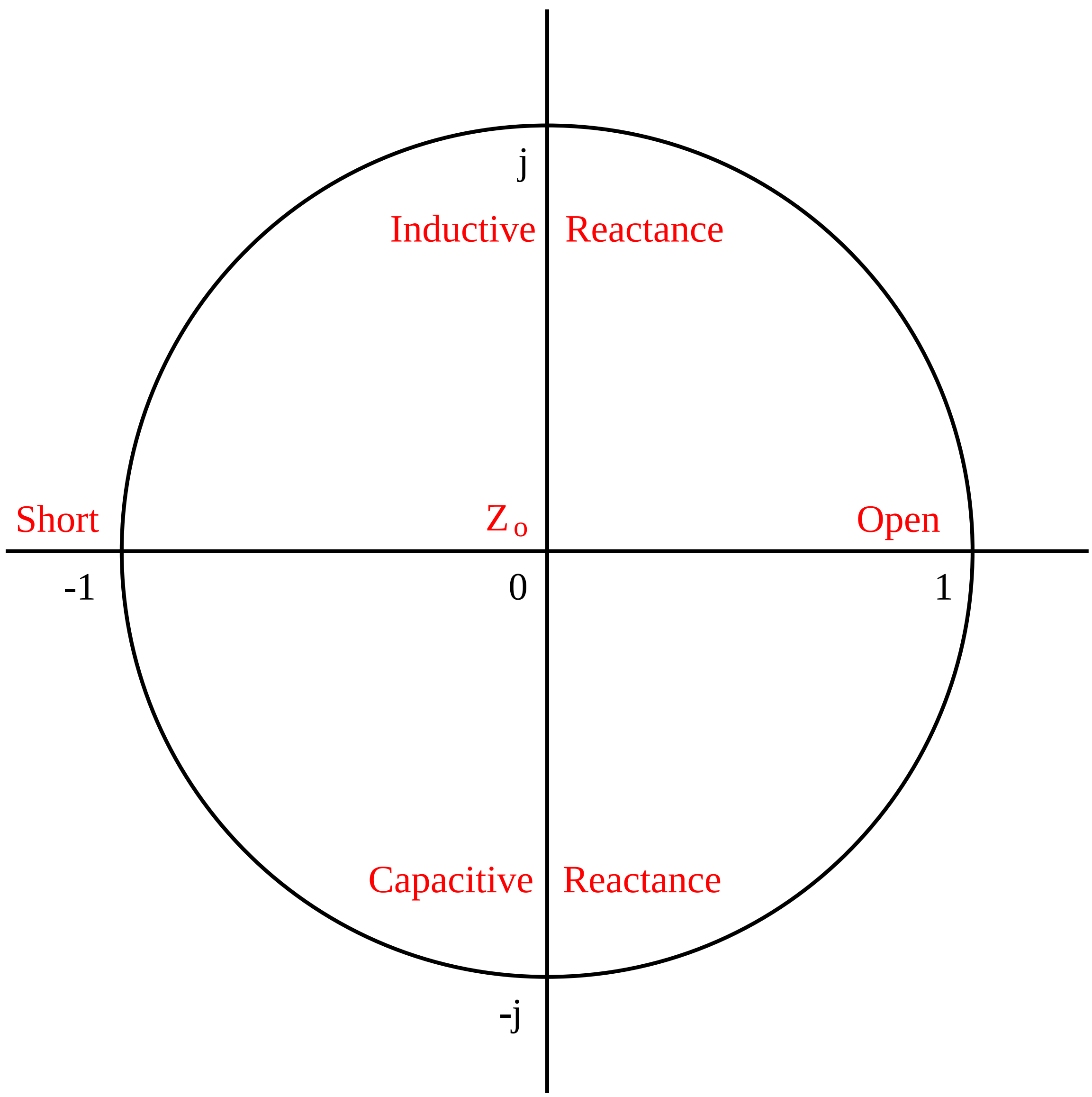
Reflections

Polar plot of $\Gamma = \frac{b_i}{a_i}$



Reflections

Polar plot of $\Gamma = \frac{b_i}{a_i}$



Johnson Noise

- Power generated by a resistor at temperature T is

$$P = k_B T \Delta f$$

- Rayleigh-Jeans Limit

$$f \ll \frac{k_B T}{h}$$

- Like a 1-d black body
- Only created in lossy circuits
- R affects how the noise couples into the circuit

Noise Waves

- Real devices generate noise

$$\mathbf{b} = \mathcal{S}\,\mathbf{a} + \mathbf{c}$$

- $\mathbf{c}$ is random, but we know the spectral density

$$\overline{|c_i|^2}$$

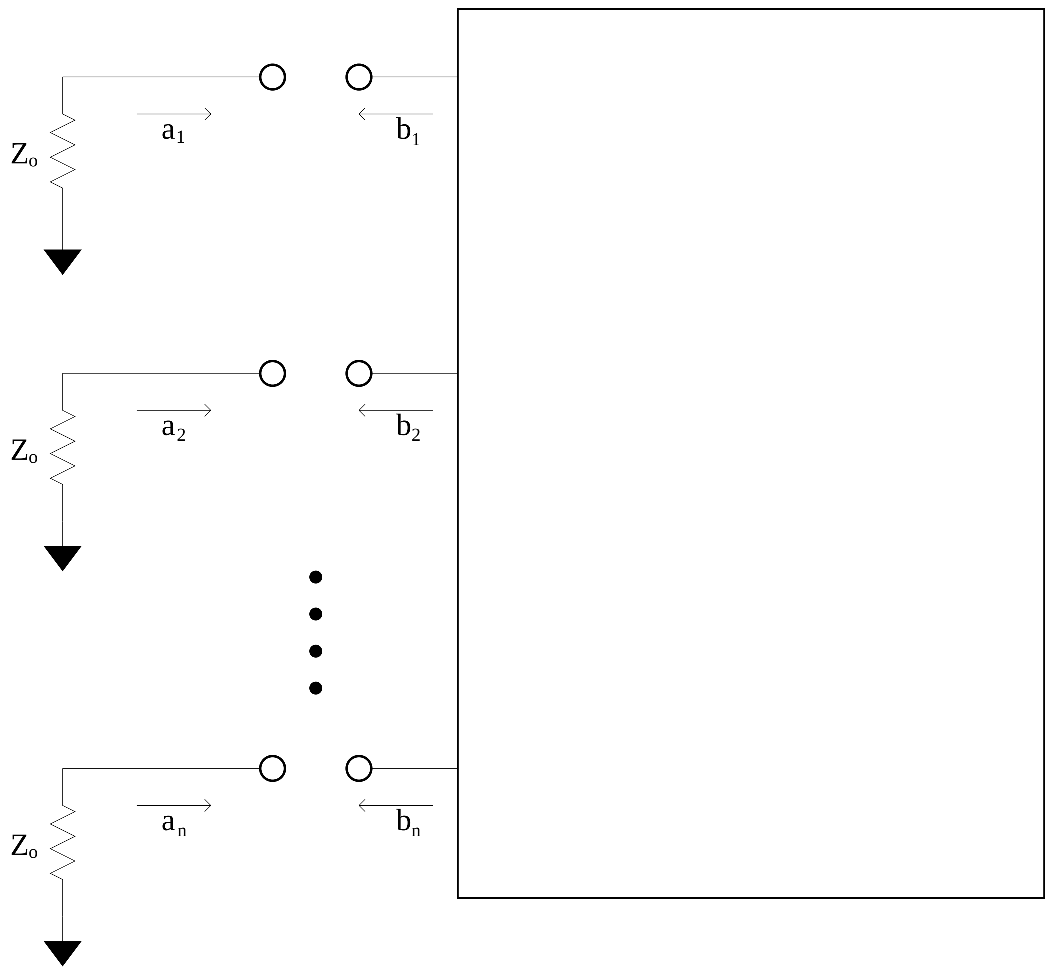
- Be careful adding noise waves

$$P = \overline{|c_1 + c_2|^2} = \overline{|c_1|^2} + \overline{|c_2|^2} + 2\,Re[\overline{c_1 c_2{}^*}]$$

- The noise wave correlation matrix is

$$\mathcal{C} \equiv \overline{\mathbf{c}\mathbf{c}^\dagger}$$

Noise Waves



Noise Matrix for Passive Circuits

- Have passive linear circuit with n ports
- Terminate each port with impedance $Z = Z_o$
- Circuit is in thermal equilibrium at temperature T
- Uncorrelated Johnson noise from the terminators, so

$$\overline{|a_i|^2} = k_B T$$

$$\overline{a_i a_j^*} = 0 \quad (i \neq j)$$

$$\overline{a\, a^\dagger} = k_B T\, \mathcal{I}$$

- Thermodynamic equilibrium requires

$$\overline{|b_i|^2} = \overline{|a_i|^2}$$

$$\overline{b_i b_j^*} = 0 \quad (i \neq j)$$

$$\overline{b\, b^\dagger} = k_B T\, \mathcal{I}$$

- Can now compute $\mathcal{C}$ from the scattering relation

$$\mathbf{b} = \mathcal{S}\, \mathbf{a} + \mathbf{c}$$

$$\overline{\mathbf{b}\mathbf{b}^\dagger} = \mathcal{S}\overline{\mathbf{a}\mathbf{a}^\dagger}\mathcal{S}^\dagger + \overline{\mathbf{c}\mathbf{c}^\dagger}$$

$$k_B T \mathcal{I} = k_B T \mathcal{S}\mathcal{S}^\dagger + \mathcal{C}$$

$$\mathcal{C} = k_B T(\mathcal{I} - \mathcal{S}\mathcal{S}^\dagger)$$

Noise in Passive Circuits

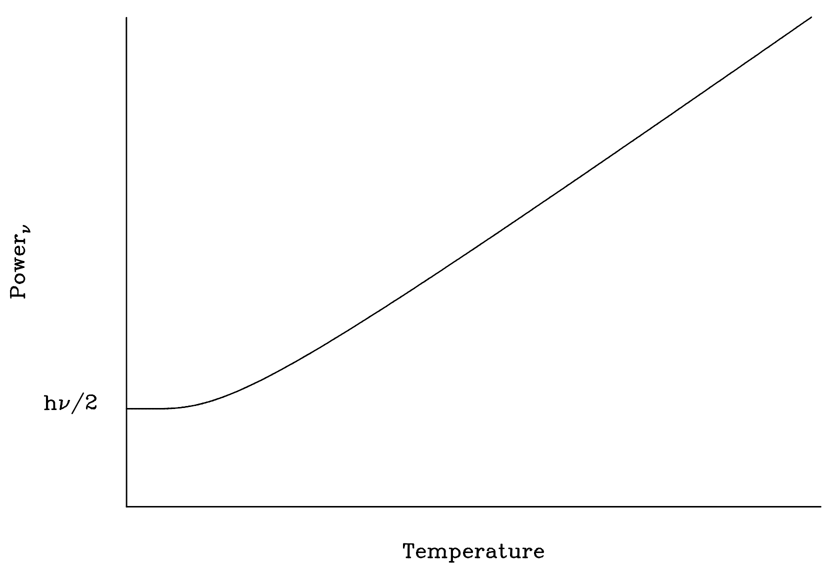
- Depends only on $\mathcal{S}$ and temperature

$$\mathcal{C} = \frac{h\nu}{2} \coth\left(\frac{h\nu}{2k_B T}\right) \left(\mathcal{I} - \mathcal{S}\mathcal{S}^\dagger\right)$$

- Hyperbolic cotangent adds quantum noise for low T:

$$\frac{h\nu}{2} \coth\left(\frac{h\nu}{2k_B T}\right) \approx k_B T \quad \text{if} \quad T \gg \frac{h\nu}{2k_B}$$

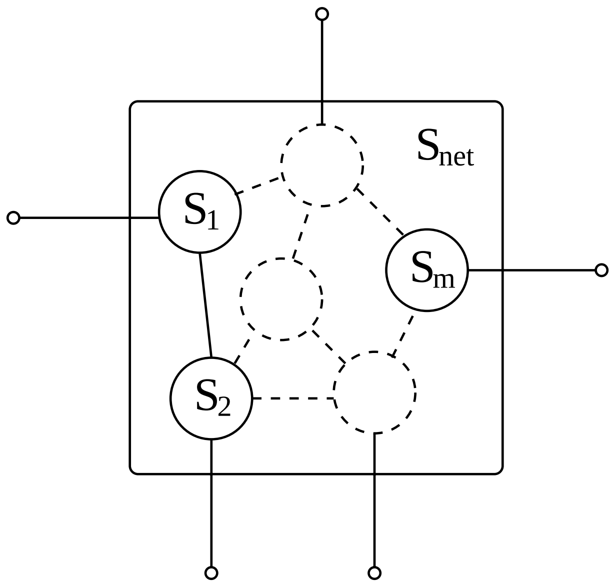
$$\frac{h\nu}{2} \coth\left(\frac{h\nu}{2k_B T}\right) \approx \frac{h\nu}{2} \quad \text{if} \quad T \ll \frac{h\nu}{2k_B}$$



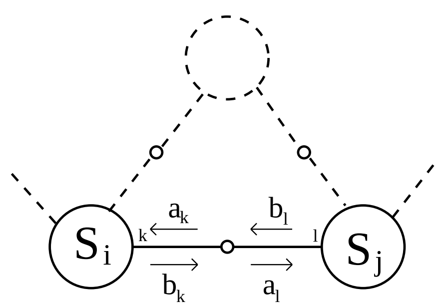
Supermix sdata class

```
// ***********************************************************
//
// class sdata
//
// sdata holds linear circuit data, including scattering
// matrix, noise correlation matrix, and source vector.
//
// ***********************************************************
class sdata
{
protected:
  double z_norm;   // what characteristic impedance was used

public:
  Matrix S;   // scattering matrix
  Matrix C;   // noise correlation matrix
  Vector B;   // source vector, in square root of power

  int size() const;

  // convert sdata values to "engineering" forms:

  // Returns S parameter in dB for in -> out
  double SdB(int out, int in) const;

  // Returns noise temperature referred to input
  double tn(int out, int in) const;

  // Returns noise figure referred to input
  double NF(int out, int in) const;

  // Calculates noise matrix for a passive element using the
  // scattering matrix, temperature, and frequency.
  sdata & passive_noise(double freq, double temp) ;
};
```

Connecting Linear Circuits



$$\begin{pmatrix} \mathbf{b_1} \\ \mathbf{b_2} \\ \vdots \\ \mathbf{b_m} \end{pmatrix} = \begin{pmatrix} \mathcal{S}_1 & 0 & .. & 0 \\ 0 & \mathcal{S}_2 & .. & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & .. & \mathcal{S}_m \end{pmatrix} \begin{pmatrix} \mathbf{a_1} \\ \mathbf{a_2} \\ \vdots \\ \mathbf{a_m} \end{pmatrix} + \begin{pmatrix} \mathbf{c_1} \\ \mathbf{c_2} \\ \vdots \\ \mathbf{c_m} \end{pmatrix}$$



$$\mathbf{b} = \Gamma \mathbf{a}$$

$\mathcal{S}$ Matrix for Connected Circuits

- Start with m independent scattering relations

$$\begin{pmatrix} \mathbf{b_1} \\ \mathbf{b_2} \\ \vdots \\ \mathbf{b_m} \end{pmatrix} = \begin{pmatrix} \mathcal{S}_1 & 0 & .. & 0 \\ 0 & \mathcal{S}_2 & .. & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & .. & \mathcal{S}_m \end{pmatrix} \begin{pmatrix} \mathbf{a_1} \\ \mathbf{a_2} \\ \vdots \\ \mathbf{a_m} \end{pmatrix} + \begin{pmatrix} \mathbf{c_1} \\ \mathbf{c_2} \\ \vdots \\ \mathbf{c_m} \end{pmatrix}$$

- Rearrange to separate external and internal ports

$$\begin{pmatrix} \mathbf{b_e} \\ \mathbf{b_i} \end{pmatrix} = \begin{pmatrix} \mathcal{S}_{ee} & \mathcal{S}_{ei} \\ \mathcal{S}_{ie} & \mathcal{S}_{ii} \end{pmatrix} \begin{pmatrix} \mathbf{a_e} \\ \mathbf{a_i} \end{pmatrix} + \begin{pmatrix} \mathbf{c_e} \\ \mathbf{c_i} \end{pmatrix}$$

- Write port connections as a matrix relation

$$\mathbf{b_i} = \Gamma \mathbf{a_i}$$

- Write the desired (sourceless) scattering relation

$$\mathbf{b_e} = \mathcal{S}_{net}\, \mathbf{a_e}$$

- Solve for $\mathcal{S}_{net}$

$$\mathcal{S}_{net} = \mathcal{S}_{ee} + \mathcal{S}_{ei}(\Gamma - \mathcal{S}_{ii})^{-1}\mathcal{S}_{ie}$$

Noise Matrix for Connected Circuits

- Start as before, except $\mathbf{a_e} = 0$ and $\mathbf{c_{net}} = \mathbf{b_e}$

$$\begin{pmatrix} \mathbf{c_{net}} \\ \mathbf{b_i} \end{pmatrix} = \begin{pmatrix} \mathcal{S}_{ee} & \mathcal{S}_{ei} \\ \mathcal{S}_{ie} & \mathcal{S}_{ii} \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{a_i} \end{pmatrix} + \begin{pmatrix} \mathbf{c_e} \\ \mathbf{c_i} \end{pmatrix}$$

- Write port connections as a matrix relation

$$\mathbf{b_i} = \Gamma \mathbf{a_i}$$

- Solve for $\mathbf{c_{net}}$

$$\mathbf{c_{net}} = \mathcal{S}_{ei}(\Gamma - \mathcal{S}_{ii})^{-1}\mathbf{c_i} + \mathbf{c_e}$$

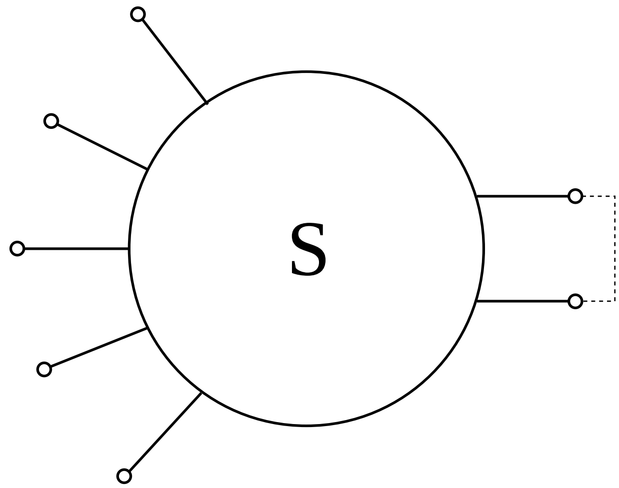
- Rearrange the input noise matrices to create $\mathcal{C}_s$

$$\mathcal{C}_s \equiv \begin{pmatrix} \overline{\mathbf{c_e}\mathbf{c_e}^\dagger} & \overline{\mathbf{c_e}\mathbf{c_i}^\dagger} \\ \overline{\mathbf{c_i}\mathbf{c_e}^\dagger} & \overline{\mathbf{c_i}\mathbf{c_i}^\dagger} \end{pmatrix}$$

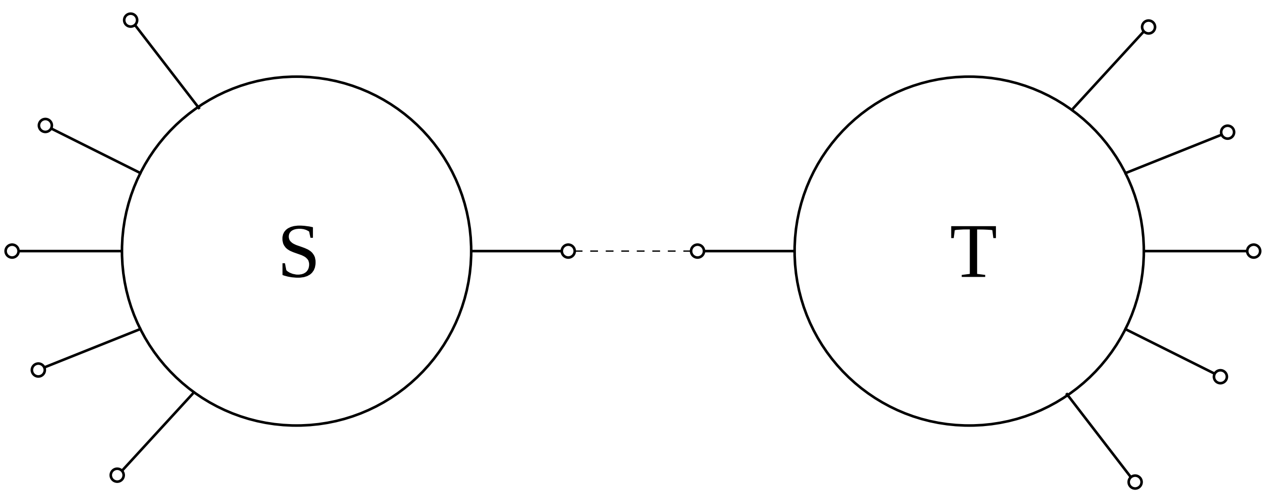
- Compute $\mathcal{C}_{net} = \overline{\mathbf{c_{net}}\,\mathbf{c_{net}}^\dagger}$

$$\mathcal{C}_{net} = [\mathcal{I}\,|\,\mathcal{S}_{ei}(\Gamma - \mathcal{S}_{ii})^{-1}]\,\mathcal{C}_s\,[\mathcal{I}\,|\,\mathcal{S}_{ei}(\Gamma - \mathcal{S}_{ii})^{-1}]^\dagger$$

Connection Types



Intra-Connection



Inter-Connection

Subnet Growth in Supermix